Nail H. Ibragimov

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Paper 1

Infinitesimal method in the theory of invariants of algebraic and differential equations

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Invited lecture at 1996 Annual Congress of South African Mathematical Society.

1 Preliminaries

1.1 Introduction

Invention of both algebraic and differential invariants can be dated by 1773 when J.L. Lagrange noted the invariance of the discriminant of the general binary quadratic form under special linear transformations of the variables (with the determinant 1), and simultaneously P.S. Laplace introduced his renowned invariants of linear partial differential equations of the second order.

It took another 70 years before G. Boole in 1841-42 generalized Lagrange's incidental observation to rational homogeneous functions (of many variables and of an arbitrary order) with the discovery of so-called *covariants* with respect to arbitrary linear transformations. Boole's success provided A. Calyley with an incentive to begin in 1845 the systematic development of a new approach to linear transformations [10]. His general theory of algebraic forms and their invariants became one of the dominating fields of pure mathematics in the 19th century.

Differential invariants were seriously considered in the 1870's by J. Cockle [12], [13], E. Laguerre [72], [73] and G.H. Halphen [24] (see also [20] and the references therein). Valuable comments on group theoretic background of differential invariants are to be found in [79].

Regretfully, many of these results are forgotten now and all the direction seems antiquated. However the recent Lie group analysis of initial value problems reveals [36] that differential invariants furnish a powerful tool for tackling complicated problems, e.g. when dealing with Huygens' principle, Riemann's method, etc.

The purpose of this paper is to outline methods from infinitesimal calculus of algebraic and differential invariants and to sketch some new results.

1.2 Algebraic invariants

It is advantageous, for successive calculation of invariants, to write algebraic and linear ordinary differential equations in a *standard form* involving the binomial coefficients.

The standard form of the general algebraic equation of the nth degree is

$$\begin{aligned} P_n(x) \equiv C_0x^n + nC_1x^{n-1} + \frac{n(n-1)}{2!}\,C_2x^{n-2} + \cdots \\ + \frac{n!}{(n-k)!\,k!}\,C_kx^{n-k} + \cdots + nC_{n-1}x + C_n = 0. \end{aligned} \tag{1.1}$$

An *equivalence transformation* of (1.1) is an invertible transformation of x mapping every equation (1.1) of the nth degree into an algebraic equation of the same degree.

Proposition 1.1. The most general group of equivalence transformations of the equations (1.1) is provided by the linear-rational transformations

$$\bar{x} = \frac{ax+\varepsilon}{b+\delta x} \tag{1.2}$$

subject to the invertibility condition

$$ab - \varepsilon\delta \neq 0. \tag{1.3}$$

An *invariant* of the equations (1.1) is a function F of the coefficients* C_i unalterable under the equivalence transformations (1.2):

$$F(C_0, C_1, \ldots, C_n) = F(\overline{C}_0, \overline{C}_1, \ldots, \overline{C}_n), \tag{1.4}$$

*Boole's *covariants*, unlike the invariants, are functions of both the coefficients and the variables.

where $\overline{C}_i$ denote the coefficients of the algebraic equation obtained from (1.1) by the transformations (1.2). In what follows, we will encounter invariants of subgroups of equivalence groups, and following Laguerre's suggestion [73] and in accordance with A. Cayley's theory of invariants, we will call them *semi-invariants* of the equations in question. Furthermore, certain invariance properties are represented by *invariant equations,*

$$H_\nu(C_0, C_1, \ldots, C_n) = 0, \quad \nu = 1, 2, \ldots, \tag{1.5}$$

where the functions H_ν are not necessarily invariants.

One encounters invariants of algebraic equations, e.g. when one applies a linear transformation of the variable x to the equation (1.1) so as to annul the term next to the highest. After this transformation, the remaining coefficients of the transformed $\overline{P}_n(\bar{x})$ are given as rational functions of C_i. For example, the quadratic equation

$$P_2(x) \equiv C_0x^2 + 2C_1x + C_2 = 0 \tag{1.6}$$

is transformed to an equation lacking the second term,

$$\overline{P}_2(\bar{x}) \equiv \bar{x}^2 + H_1 = 0,$$

e.g. by the substitution $\bar{x} = C_0x + C_1$. Then H_1 is the *discriminant*,

$$H_1 = C_0C_2 - C_1^2. \tag{1.7}$$

Likewise, one can apply the substitution $\bar{x} = C_0x + C_1$ to transform the cubic equation

$$P_3(x) \equiv C_0x^3 + 3C_1x^2 + 3C_2x + C_3 = 0 \tag{1.8}$$

to an equation lacking the second term,

$$\overline{P}_3(\bar{x}) \equiv \bar{x}^3 + 3H_1\bar{x} + H_2 = 0,$$

where the first coefficient H_1 is again given by (1.7), and the second one is

$$H_2 = C_0^2C_3 - 3C_0C_1C_2 + 2C_1^3. \tag{1.9}$$

The functions (1.7), (1.9) are the same for all $n > 3$. The vanishing of both H_1 and H_2 defines a system of invariant equations (1.5) and provides the necessary and sufficient condition for the equation (1.8) to have three equal roots.

E.W. Tschirnhausen in 1683 applied to the equation (1.1) more general transformations* to eliminate the terms of the degrees $n-1, n-2$, etc. (see, e.g. [4], Chap. XII). It was noticed, however, by G.W. Leibnitz that the application of Tschirnhausen's transformation trying to get rid also of the $(n-4)$th term requires the solution of equations more complicated than the original one.

1.3 Invariants of linear differential equations

The standard form of the general linear homogeneous ordinary differential equation of the nth order with (regular) variable coefficients $c_i(x)$ is

$$\begin{aligned} L_n(y) \equiv y^{(n)} + nc_1(x)y^{(n-1)} + \frac{n(n-1)}{2!}\, c_2(x)y^{(n-2)} + \cdots \\ + \frac{n!}{(n-k)!\,k!}\, c_k(x)y^{(k)} + \cdots + nc_{n-1}(x)y' + c_n(x)y = 0. \end{aligned} \tag{1.10}$$

An equivalence transformation of the equations (1.10) is an invertible transformation of the independent and dependent variables, x and y, preserving the order n of any equation (1.10) as well as its linearity and homogeneity.

Proposition 1.2. The most general group of equivalence transformations of the equations (1.10) is an infinite group composed of linear transformations of the dependent variable:

$$y = \sigma(x)z, \quad \sigma(x) \neq 0, \tag{1.11}$$

and invertible changes of the independent variable:

$$\bar{x} = \phi(x), \quad \phi'(x) \neq 0, \tag{1.12}$$

where $\sigma(x)$ and $\phi(x)$ are arbitrary n times continuously differentiable functions.

Differential invariants† of equations (1.10) were found in the problem of practical solution of differential equations by reducing them to equivalent but readily integrable forms ([13], [72], [73], [24]) or by using relations

*They can be reduced to polynomial substitutions $\bar{x} = x^r + A_1x^{r-1} + \cdots + A_r,\ r < n$, with unknown coefficients A_i to be determined from the condition that an equation (1.1) becomes $B_0y^n + B_1y^{n-1} + B_2y^{n-2} + \cdots + B_n = 0$ with $B_1 = 0, B_2 = 0, \ldots$. The inverse of this substitution is not single-valued and hence Tschirnhausen's transformations are not equivalence transformations.

†More precisely, they are differential invariants of the subgroup (1.11) of the general equivalence given by (1.11)-(1.12), and they are termed in what follows *semi-invariants*.

between solutions of a given equation [86]. For example, the second-order equations

$$L_2(y) \equiv y'' + 2c_1(x)y' + c_2(x)y = 0 \tag{1.13}$$

have one differential semi-invariant,

$$h_1 = c_2 - c_1^2 - c_1', \tag{1.14}$$

the third-order equations have two semi-invariants, namely (1.14) and

$$h_2 = c_3 - 3c_1c_2 + 2c_1^3 - c_1''. \tag{1.15}$$

An astonishing similarity of the *semi-invariants* H_ν and h_ν of algebraic and differential equations, respectively, is deeper than a transparent likeness of the formulae (1.7), (1.9) and (1.14), (1.15). This similarity inspired profound investigations of the theory of invariants by brilliant mathematicians of the 19th century (e.g. [12], [13], [72], [73], [24], [86], [25], [20]).

An analog of Tschirnhausen's transformation for differential equations (1.10) was also discovered in the 19th century. Namely, J. Cockle [13] (for $n = 3$) and E. Laguerre [73] (for the general equation) independently showed that the two terms of orders next below the highest can be simultaneously removed in any equation (1.10) by a proper combination of transformations (1.11) and (1.12). See also [20], p. 403.

Some 100 years prior to the aforementioned historical events P.S. Laplace, in his fundamental memoir [75] dedicated to integration of linear partial differential equations, discovered *inter alia* two invariants:

$$h = a_\xi + ab - c, \quad k = b_\eta + ab - c, \tag{1.16}$$

for the general hyperbolic second-order equations with two independent variables,

$$u_{\xi\eta} + a(\xi,\eta)u_\xi + b(\xi,\eta)u_\eta + c(\xi,\eta)u = 0. \tag{1.17}$$

Here, as usual, u_ξ etc. denote partial derivatives.

Setting $\xi = \eta = x,\ u(x,x) = y(x),\ a(x,x) = b(x,x) = c_1(x),\ c(x,x) = c_2(x)$ in (1.17) and (1.16), one obtains the second-order ordinary differential equation (1.13) and its invariant (1.14):

$$h = k = c_1' + c_1^2 - c_2 \equiv -h_1\,.$$

Proposition 1.3. The most general group of equivalence transformations of the equations (1.17) is an infinite group composed of linear transformations of the dependent variable:

$$u = \sigma(\xi,\eta)v, \quad \sigma(\xi,\eta) \neq 0, \tag{1.18}$$

and invertible changes of the independent variables of the form:

$$\bar{\xi} = \phi(\xi), \quad \bar{\eta} = \psi(\eta), \tag{1.19}$$

where σ, ϕ, ψ are arbitrary regular functions.

The Laplace invariants (1.16) are invariant only with respect to the transformations (1.18). Hence h and k are semi-invariants* of Eq. (1.17). It took almost another 200 years before L.V. Ovsyannikov [90] found two proper invariants, namely

$$p = \frac{k}{h}, \quad \text{and} \quad q = \frac{1}{h}\frac{d^2 \ln|h|}{d\xi d\eta}\,, \tag{1.20}$$

that are invariant under the general group of the equivalence transformations (1.18) - (1.19).

2 Outline of the method via algebraic equations

2.1 Linear form of equivalence transformations

Equation (1.1) is rewritten, by setting $x = u/v$ and multiplying the resulting equation by v^n, in the homogeneous form:

$$\begin{aligned} Q_n(u,v) \equiv C_0u^n + nC_1u^{n-1}v + \frac{n(n-1)}{2!}\,C_2u^{n-2}v^2 + \cdots \\ + \frac{n!}{(n-k)!\,k!}\,C_ku^{n-k}v^k + \cdots + nC_{n-1}uv^{n-1} + C_nv^n = 0. \end{aligned} \tag{2.1}$$

Then (1.2) reduces to the linear mapping $\bar{u} = au + \varepsilon v, \ \bar{v} = bv + \delta u$. This simplifies calculations, e.g. makes Proposition 1.1 self-evident. We let $a = \mathrm{e}^\alpha, b = \mathrm{e}^\beta$ and consider the *infinitesimal equivalence transformation*

$$\bar{u} \approx u + (\alpha u + \varepsilon v), \quad \bar{v} \approx v + (\beta v + \delta u).$$

Its inverse, written in the first order of precision with respect to the small parameters $\alpha, \beta, \varepsilon, \delta$, has the form:

$$u \approx \bar{u} - (\alpha\bar{u} + \varepsilon\bar{v}), \quad v \approx \bar{v} - (\beta\bar{v} + \delta\bar{u}). \tag{2.2}$$

**Author's note to this 2009 edition:* I learned in 2006 from Louise Petrén's Thesis [95] that the quantities (1.16) were earlier known to L. Euler [19]. Therefore one can call them more accurately the *Euler-Laplace semi-invariants.*

2.2 Quadratic equation

Substitution of (2.2) into the quadratic equation

$$Q_2(u,v) \equiv C_0 u^2 + 2C_1 uv + C_2 v^2 = 0 \tag{2.3}$$

transforms it to $\overline{Q}_2(\bar{u},\bar{v}) \equiv \overline{C}_0 \bar{u}^2 + 2\overline{C}_1 \bar{u}\bar{v} + \overline{C}_2 \bar{v}^2 = 0$, where

$$\begin{aligned} \overline{C}_0 &\approx C_0 - 2(\alpha C_0 + \delta C_1), \\ \overline{C}_1 &\approx C_1 - (\alpha C_1 + \beta C_1 + \varepsilon C_0 + \delta C_2), \\ \overline{C}_2 &\approx C_2 - 2(\beta C_2 + \varepsilon C_1). \end{aligned} \tag{2.4}$$

Equations (2.4) provide the infinitesimal transformations of the 4-parameter group G_4^2 with the basic generators

$$X_1 = 2C_0 \frac{\partial}{\partial C_0} + C_1 \frac{\partial}{\partial C_1}\,, \quad X_2 = C_1 \frac{\partial}{\partial C_1} + 2C_2 \frac{\partial}{\partial C_2}\,,$$
$$X_3 = C_0 \frac{\partial}{\partial C_1} + 2C_1 \frac{\partial}{\partial C_2}\,, \quad X_4 = 2C_1 \frac{\partial}{\partial C_0} + C_2 \frac{\partial}{\partial C_1}\,.$$

The invariants $F(C_0, C_1, C_2)$ of G_4^2 (see (1.4)) are determined by the equations $X_s(F) = 0$ $(s = 1, \ldots, 4)$. The latter have $3 - r_*$ functionally independent solutions with r_* denoting the *generic rank* of the matrix A associated with the operators X_s:

$$A = \left\| \begin{array}{ccc} 2C_0 & C_1 & 0 \\ 0 & C_1 & 2C_2 \\ 0 & C_0 & 2C_1 \\ 2C_1 & C_2 & 0 \end{array} \right\|. \tag{2.5}$$

Since here $r_* = 3$, there are no invariants. Furthermore, it follows from $r_* = 3$ that the group G_4^2 has no *regular* invariant equations (1.5) as well, but it may have *singular* ones.

According to the Lie group analysis, a singular invariant equation

$$H(C_0, C_1, C_2) = 0$$

is found (see, e.g. [36], Chapter 1) by imposing the condition

$$rank A\big|_{H=0} < 3$$

on the elements of the matrix (2.5) and then testing the infinitesimal invariance criterion $X_s(H)\Big|_{H=0} = 0, \ s = 1, \ldots, 4$. Implementation of this algorithm shows that there is only one singular invariant equation, namely

$$H_1 \equiv C_0 C_2 - C_1^2 = 0, \tag{2.6}$$

where H_1 is the discriminant (1.7).

2.3 Cubic equation

Likewise, by substituting (2.2) into the cubic form

$$Q_3(u,v) = C_0u^3 + 3C_1u^2v + 3C_2uv^2 + C_3v^3,$$

one arrives at the 4-parameter group G_4^3 generated by

$$X_1 = 3C_0\frac{\partial}{\partial C_0} + 2C_1\frac{\partial}{\partial C_1} + C_2\frac{\partial}{\partial C_2}\,,$$

$$X_2 = C_1\frac{\partial}{\partial C_1} + 2C_2\frac{\partial}{\partial C_2} + 3C_3\frac{\partial}{\partial C_3}\,,$$

$$X_3 = C_0\frac{\partial}{\partial C_1} + 2C_1\frac{\partial}{\partial C_2} + 3C_2\frac{\partial}{\partial C_3}\,,$$

$$X_4 = 3C_1\frac{\partial}{\partial C_0} + 2C_2\frac{\partial}{\partial C_1} + C_3\frac{\partial}{\partial C_2}\,,$$

and at the following associated matrix:

$$A = \left\| \begin{matrix} 3C_0 & 2C_1 & C_2 & 0 \\ 0 & C_1 & 2C_2 & 3C_3 \\ 0 & C_0 & 2C_1 & 3C_2 \\ 3C_1 & 2C_2 & C_3 & 0 \end{matrix} \right\|.$$

Here again the generic rank r_* of A is equal to the number of the transformed quantities C_i, i.e. $r_* = 4$. Hence G_4^3 has neither invariants nor regular invariant equations.

But it has singular invariant equations of two types. One of them is obtained by letting $rank A = 3$, i.e. $det A = 0$. It is given by

$$\Delta \equiv (C_0C_3)^2 - 6C_0C_1C_2C_3 + 4C_0C_2^3 - 3(C_1C_2)^2 + 4C_1^3C_3 = 0, \qquad (2.7)$$

where Δ is known as the *discriminant* of the cubic equation (see [4]).

The second type of singular invariant equations is obtained by letting $rank A = 2$, i.e. by annulling the minors of all elements of A, and applying the invariance test. It follows:

$$H \equiv C_0C_2 - C_1^2 = 0, \quad F \equiv C_0^2C_3 - C_1^3 = 0. \qquad (2.8)$$

Noting that here $H = H_1$ and $F = H_2 + 3C_1H_1$ with H_1 and H_2 given by (1.7) and (1.9), one can rewrite (2.8) in the equivalent form $H_1 = 0,\ H_2 = 0.$

3 Application to linear ODEs

I illustrate the method by the third-order ordinary differential equation

$$L_3(y) \equiv y''' + 3c_1(x)y'' + 3c_2(x)y' + c_3(x)y = 0. \tag{3.1}$$

3.1 Semi-invariants

Let us implement the transformation (1.11) by letting

$$\sigma(x) = 1 - \varepsilon\eta(x)$$

with a small parameter ε. Eq. (3.1) becomes

$$z''' + 3\overline{c}_1(x)z'' + 3\overline{c}_2(x)z' + \overline{c}_3(x)z = 0,$$

where

$$\begin{aligned} &\overline{c}_1 \approx c_1 - \varepsilon\eta',\\ &\overline{c}_2 \approx c_2 - \varepsilon(\eta'' + 2c_1\eta'),\\ &\overline{c}_3 \approx c_3 - \varepsilon(\eta''' + 3c_1\eta'' + 3c_2\eta'). \end{aligned} \tag{3.2}$$

Equations (3.2) provide the following group generator prolonged to all derivatives of $c_i(x)$:

$$\begin{aligned} X_\eta =& \eta'\frac{\partial}{\partial c_1} + \left(\eta'' + 2c_1\eta'\right)\frac{\partial}{\partial c_2} + \left(\eta''' + 3c_1\eta'' + 3c_2\eta'\right)\frac{\partial}{\partial c_3}\\ &+ \eta''\frac{\partial}{\partial c_1'} + \left(\eta''' + 2c_1\eta'' + 2c_1'\eta'\right)\frac{\partial}{\partial c_2'}\\ &+ \left(\eta^{(iv)} + 3c_1\eta''' + 3c_2\eta'' + 3c_1'\eta'' + 3c_2'\eta'\right)\frac{\partial}{\partial c_3'} + \cdots . \end{aligned}$$

Definition 1.1. *Semi-invariants* of Equation (3.1) are differential invariants of the infinitesimal transformation (3.2). Namely, they are functions

$$h = h(c, c', c'', \ldots)$$

of the coefficients $c = (c_1, c_2, c_3)$ of Equation (3.1) and their derivatives

$$c' = (c_1', c_2', c_3'), \quad c'' = (c_1'', c_2'', c_3''), \ldots$$

of any finite order satisfying the invariance condition

$$X_\eta(h) = 0$$

for any function $\eta(x)$.

Lemma 1.1. Equation (3.1) has two independent semi-invariants, namely:

$$
\begin{aligned}
h &= c_2 - c_1^2 - c_1', \\
f &= c_3 - 3c_1c_2 + 2c_1^3 + 2c_1c_1' - c_2'.
\end{aligned}
\tag{3.3}
$$

Any semi-invariant is a function of h, f and their derivatives.

Proof. Let $h = h(c)$ and

$$
X_\eta(h) \equiv \eta' \frac{\partial h}{\partial c_1} + \big(\eta'' + 2c_1\eta'\big)\frac{\partial h}{\partial c_2} + \big(\eta''' + 3c_1\eta'' + 3c_2\eta'\big)\frac{\partial h}{\partial c_3} = 0. \tag{3.4}
$$

Since the function $\eta(x)$ is arbitrary, there are no relations between its derivatives. Therefore the equation (3.4) splits into the following three equations obtained by annulling separately the terms with η''', η'' and η' :

$$
\frac{\partial h}{\partial c_3} = 0, \quad \frac{\partial h}{\partial c_2} + 3c_1\frac{\partial h}{\partial c_3} = 0, \quad \frac{\partial h}{\partial c_1} + 2c_1\frac{\partial h}{\partial c_2} + 3c_2\frac{\partial h}{\partial c_3} = 0.
$$

Whence $h =$ const., i.e. *there no differential invariants of the order* 0.

Likewise, substituting

$$
h = h(c, c')
$$

into the equation

$$
X_\eta(h) = 0
$$

and nullifying the term with $\eta^{(iv)}$ we obtain that

$$
\frac{\partial h}{\partial c_3'} = 0.
$$

Furthermore, the terms with η''', η'', η' give three linear partial differential equations for the function h of five variables, $c_1, c_2, c_3, c_1', c_2'$. These equations have precisely two functionally independent solutions, e.g. (3.3).

It can be easily verified that the equation $X_\eta(h) = 0$ has precisely four functionally independent solutions involving c, c', c''. Since h, f together with their first derivatives h', f' provide four functionally independent solutions of this type, the lemma is proved for differential invariants of the second order. The iteration completes the proof.

Remark 1.1. The basic semi-invariants (3.3) can be replaced by h_1, h_2 given by Eqs. (1.14)-(1.15). Indeed, $h_1 = h$ and $h_2 = f + h'$.

3.2 Invariants

Definition 1.2. *Invariants* of Equation (3.1) are the semi-invariants $J = J(h, f, h', f', \ldots)$ satisfying the additional invariance condition with respect to the transformation (1.12).

Let us find J by implementing the infinitesimal transformation (1.12),

$$\bar{x} \approx x + \varepsilon\xi(x).$$

In the notation

$$y(x) = \overline{y}(\bar{x}), \quad \frac{d\overline{y}}{d\bar{x}} = \overline{y'},$$

the chain rule implies:

$$\begin{aligned}
y' &\approx (1 + \varepsilon\xi')\overline{y'},\\
y'' &\approx (1 + 2\varepsilon\xi')\overline{y''} + \varepsilon\overline{y'}\xi'',\\
y''' &\approx (1 + 3\varepsilon\xi')\overline{y'''} + 3\varepsilon\overline{y''}\xi'' + \varepsilon\overline{y'}\xi'''.
\end{aligned}$$

Consequently Equation (3.1) becomes

$$\overline{y'''} + 3\overline{c}_1\overline{y''} + 3\overline{c}_2\overline{y'} + \overline{c}_3\overline{y} = 0,$$

where

$$\begin{aligned}
\overline{c}_1 &\approx c_1 + \varepsilon(\xi'' - c_1\xi'),\\
\overline{c}_2 &\approx c_2 + \varepsilon(\frac{1}{3}\xi''' + c_1\xi'' - 2c_2\xi'),\\
\overline{c}_3 &\approx c_3 - 3\varepsilon c_3\xi'.
\end{aligned} \tag{3.5}$$

The corresponding group generator (prolonged to derivatives of $c(x)$) is written in the space of the semi-invariants (3.3) as follows:

$$\begin{aligned}
X_\xi = \xi\frac{\partial}{\partial x} &- \big(\frac{2}{3}\xi''' + 2h\xi'\big)\frac{\partial}{\partial h} - \big(\frac{1}{3}\xi^{(iv)} + h\xi'' + 3f\xi'\big)\frac{\partial}{\partial f}\\
&- \big(\frac{2}{3}\xi^{(iv)} + 2h\xi'' + 3h'\xi'\big)\frac{\partial}{\partial h'} - \big(\frac{1}{3}\xi^{(v)} + h\xi''' + h'\xi'' + 3f\xi'' + 4f'\xi'\big)\frac{\partial}{\partial f'}\\
&- \big(\frac{2}{3}\xi^{(v)} + 2h\xi''' + 5h'\xi'' + 4h''\xi'\big)\frac{\partial}{\partial h''} + \cdots .
\end{aligned}$$

By applying the philosophy of the proof of Lemma 1.1 to the equation

$$X_\xi(J) = 0$$

one obtains the following results.

Theorem 1.1. The third-order differential equation (3.1) has the following singular invariant equation with respect to the group of general equivalence transformations (1.11)-(1.12):

$$\lambda \equiv h' - 2f = 0. \tag{3.6}$$

Theorem 1.2. The *least invariant* of the equation (3.1), i.e. a solution to $X_\xi(J) = 0$ involving the derivatives of h and f of the lowest order is

$$\theta = \frac{1}{\lambda^2}\left[7\left(\frac{\lambda'}{\lambda}\right)^2 - 6\frac{\lambda''}{\lambda} + 9h\right]^3, \tag{3.7}$$

where $\lambda = h' - 2f$. The higher order invariants are obtained from θ by means of Lie-Tresse's invariant differentiation. Any invariant J of an arbitrary order is a function of the least invariant θ and its invariant derivatives*.

Example 1.1. Equation (3.1) is equivalent to the equation

$$y''' = 0$$

if and only if the equation (3.6) holds, $\lambda = 0$ (see [73]).

Example 1.2. Equation (3.1) is equivalent to the equation

$$y''' + y = 0$$

if and only if $\lambda \neq 0$ but the invariant (3.7) vanishes, $\theta = 0$. This condition is satisfied, e.g. for the equation

$$y''' + \frac{y}{(kx+l)^6} = 0$$

with constants k and l not vanishing simultaneously.

4 Invariants of nonlinear ODEs

The infinitesimal method can be used for investigation of invariants of nonlinear differential equations as well. Consider an example due to S. Lie [77]. The set of nonlinear equations cubic in the first derivative,

$$y'' + a(x,y)y'^{\,3} + b(x,y)y'^{\,2} + c(x,y)y' + d(x,y) = 0, \tag{4.1}$$

**Author's note to this 2009 edition:* A discussion of invariants of Equation (3.1) in the framework of the projective differential geometry can be found in [74], Chapter I, Section 4. I thank Valentin Lychagin for drawing my attention to the book [74].

contains all second-order equations obtained from the linear equation (1.13) by arbitrary changes of the variables,

$$\bar{x} = f(x, y), \quad \bar{y} = g(x, y). \tag{4.2}$$

Furthermore, the variety of equations (4.1) with arbitrary coefficients $a(x, y)$, $b(x, y), c(x, y)$, $d(x, y)$ is invariant under the infinite group G of transformations (4.2). Consequently, G is the group of equivalence transformations of the equations (4.1).

Theorem 1.3. The system of equations

$$H \equiv 3a_{xx} - 2b_{xy} + c_{yy} - 3ac_x - 3ca_x + 2bb_x + 3da_y + 6ad_y - bc_y = 0, \tag{4.3}$$

$$K \equiv 3d_{yy} - 2c_{xy} + b_{xx} - 3ad_x - 6da_x + 3bd_y + 3db_y - 2cc_y + cb_x = 0, \tag{4.4}$$

is invariant under the group G of transformations (4.2) and specifies, among the nonlinear equations (4.1), all linearizable ones.

5 On Ovsyannikov's invariants

The calculations presented above can naturally be extended to all algebraic and linear ordinary differential equations, (1.1) and (1.10). The infinitesimal method can be used also in the case of partial differential equations. For example, its application to the linear hyperbolic equation (1.17) shows that Ovsyannikov's invariants p and q given by Eqs. (1.20) provide an essential part of a basis of invariants as stated in the following theorem*.

Theorem 1.4. A basis of invariants of the hyperbolic equations (1.17) is provided by the invariants

$$p = \frac{k}{h}, \quad I = \frac{p_x\, p_y}{h}, \quad N = \frac{1}{p_x} \frac{\partial}{\partial x} \ln \left| \frac{p_x}{h} \right|, \quad q = \frac{1}{h} \frac{\partial^2 \ln |h|}{\partial x \partial y}. \tag{5.5}$$

Any other invariant of an arbitrary order is a function of the invariants (5.5) and their derivatives obtained by the following invariant differentiations:

$$\mathcal{D}_1 = \frac{1}{p_x} D_x, \quad \mathcal{D}_2 = \frac{1}{p_y} D_y. \tag{5.6}$$

**Author's note to this 2009 edition:* The result formulated below is based on the calculations revised in 2004 and published in [43].

6 Invariants of evolutionary equations

The concept of invariants of differential equations can be useful for (linear or nonlinear) evolutionary equations. For illustration purposes, let's consider the Maxwell equations. Their evolutionary part,

$$\boldsymbol{E}_t = \nabla \times \boldsymbol{B}, \quad \boldsymbol{B}_t = -\nabla \times \boldsymbol{E}, \tag{6.1}$$

defines a particular Lie-Bäcklund group (t is a group parameter) with the infinitesimal generator

$$X = \sum_{i=1}^{3} \left((\nabla \times \boldsymbol{B})^i \frac{\partial}{\partial E^i} - (\nabla \times \boldsymbol{E})^i \frac{\partial}{\partial B^i} \right).$$

Theorem 1.5. The Lie-Bäcklund group defined by Eqs. (6.1) have the following basis of invariants ([34]):

$$\operatorname{div} \boldsymbol{E}, \quad \operatorname{div} \boldsymbol{B}. \tag{6.2}$$

All other differential invariants of an arbitrary order are functions of the basic invariants (6.2) and their successive derivatives with respect to the spatial variables $x = (x^1, x^2, x^3)$.

Theorem 1.5 guaranties the solvability of initial value problems for the *over-determined* system of generalized Maxwell equations consisting of (6.1) and of additional differential equations of the form

$$\operatorname{div} \boldsymbol{E} = 4\pi\rho(x), \quad \operatorname{div} \boldsymbol{B} = g(x), \tag{6.3}$$

where $g(x) = 0$ corresponds to Maxwell's equations. Theorem 4.1 states that the additional equations (6.3) are satisfied identically provided that an initial data,

$$\boldsymbol{E}\big|_{t=0} = \boldsymbol{E}_0(x), \quad \boldsymbol{B}\big|_{t=0} = \boldsymbol{B}_0(x), \tag{6.4}$$

satisfies the constraints (6.3)

$$\operatorname{div} \boldsymbol{E}_0 = 4\pi\rho(x), \quad \operatorname{div} \boldsymbol{B}_0 = g(x). \tag{6.5}$$

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Paper 2

Group analysis of a tumour growth model

Talk at MOGRAN-9
Moscow, 2002
(See [60], also [61])

Abstract. Recently, several mathematical models appeared in the literature for describing spread of malignant tumours. These models are given by systems of nonlinear partial differential equations containing, in general, several unknown functions of dependent variables. Determination of these unknown functions (called in Lie group analysis *arbitrary elements*) is a complicated problem that challenges researchers.

We consider a system of nonlinear partial differential equations suggested by A.J. Perumpanani et al. [94] for modelling malignant tumours spread into neighbouring tissue in the absence of cell diffusion. The authors of the model discussed its biological significance by using travelling wave solutions as well as numerical and power series methods. J.M. Stewart, P. Broadbridge and J.M. Goard [101] investigated group invariant solutions in a particular case of this model. They also generalised the model by adding diffusive terms.

In the present work, we consider the original model of A.J. Perumpanani et al. and find the equivalence Lie algebra of this model. Using some of the equivalence generators, we find additional symmetries and invariant solutions.

1 Formulation of the problem

In healthy tissue, balance is preserved between cellular reproduction and cell death. A change of DNA caused by genetic, chemical or other environmental reasons, can give rise to a malignant tumour cell which disrupts this balance and causes an uncontrolled reproduction of cells followed by infiltration into neighboring or remote tissues (metastasis).

Perumpanani et al. [94] investigated the problem of invasion of malignant cells into surrounding tissue neglecting cellular diffusion. Motivated by several important observations in tumour biology (e.g. [3]), they suggested a mathematical model for invasion by hapotaxis (i.e. directed movement that occur in response to a fixed substrate) and protease production. The model studies the averaged one-dimensional spatial dynamics of malignant cells by ignoring variations in the plane perpendicular to the direction of invasion. The model is formulated in terms of nonlinear partial differential equations as the following system:

$$\begin{aligned} u_t &= f(u) - (uc_x)_x\,, \\ c_t &= -g(c,p), \\ p_t &= h(u,c) - Kp. \end{aligned} \tag{1.1}$$

Here u, c and p, depend on time t and one space coordinate x and represent the concentrations of invasive cells, extracellular matrix (e.g. type IV collagen) and protease, respectively. To describe the dynamics of a specific biological system, the authors of the model introduced *arbitrary elements* $f(u)$, $g(c,p)$ and $h(u,c)$ that are supposed to be increasing functions of the dependent variables u, c, p. For example, the function $h(u,c)$ in last equation of the above system represents the dependence of the protease production on local concentrations of malignant cells and collagen, while the term $-Kp$ is based on the assumption that the protease decays linearly, where K is a positive constant to be determined experimentally via half-life.

Note that we use the common convention to denote by subscripts the respective differentiations, e.g. $u_t = \partial u/\partial t$.

By observing that the timescales associated with the protease production and decay are considerably shorter than for the invading cells, the model (1.1) can be reduced to the following system of two equations [94], [101]:

$$\begin{aligned} u_t &= f(u) - (uc_x)_x\,, \\ c_t &= -g(c,u), \end{aligned} \tag{1.2}$$

where $f(u)$ and $g(c,u)$ are arbitrary functions satisfying the conditions

$$f(u) > 0, \quad g_c(c,u) > 0, \quad g_u(c,u) > 0.$$

The main part of the paper [94] is dedicated to discussion of the particular case $f(u) = u(u-1)$, corresponding to a logistic production rate, and $g(c,u) = uc^2$. Another particular case, namely $g(c,u) = uh(c)$, involving two arbitrary functions $f(u)$ and $h(c)$ of one variable, is discussed in detail in [101], where the functions $f(u)$ and $h(c)$ are classified according to symmetries of the corresponding system (1.2).

Our goal is to find and employ the Lie algebra of the generators of the equivalence transformations for the model (1.2) with arbitrary functions $f(u)$ and $g(c,u)$. Note that equations (1.2) do not involve explicitly the independent variables t and x. Consequently, the system (1.2) with arbitrary functions $f(u)$ and $g(c,u)$ is invariant under the two-parameter group of translations of t and x. Recall that the Lie algebra of the maximal group admitted by the system (1.2) is termed the *principal Lie algebra* for (1.2) and is denoted $L_{\mathcal{P}}$. Thus, the algebra $L_{\mathcal{P}}$ contains at least two operators:

$$X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial x}\,\cdot \tag{1.3}$$

2 Equivalence Lie algebra

An equivalence transformation of the equations (1.2) is a change of variables $(t,x,u,c) \to (\bar{t},\bar{x},\bar{u},\bar{c})$ taking the system (1.2) into a system of the same form, generally speaking, with different functions $\bar{f}(\bar{u})$ and $\bar{g}(\bar{c},\bar{u})$. The generators of the continuous group of equivalence transformations have the form

$$Y = \xi^1\frac{\partial}{\partial t} + \xi^2\frac{\partial}{\partial x} + \eta^1\frac{\partial}{\partial u} + \eta^2\frac{\partial}{\partial c} + \mu^1\frac{\partial}{\partial f} + \mu^2\frac{\partial}{\partial g}\,, \tag{2.1}$$

where

$$\xi^i = \xi^i(t,x,u,c), \quad \eta^i = \eta^i(t,x,u,c), \quad \mu^i = \mu^i(t,x,u,c,f,g), \quad i = 1,2.$$

Recall (see [92]) that the operator (2.1) generates the equivalence group if it is admitted by the *extended* system (1.2):

$$\begin{array}{l} u_t - f + u_x c_x + u c_{xx} = 0, \quad c_t + g = 0, \\ f_t = 0,\ f_x = 0,\ f_c = 0,\ g_t = 0,\ g_x = 0. \end{array} \tag{2.2}$$

The infinitesimal invariance test for the system (2.2) requires the following prolongation of the operator (2.1):

$$\begin{aligned} \widetilde{Y} = Y \quad & + \zeta_1^1\frac{\partial}{\partial u_t} + \zeta_2^1\frac{\partial}{\partial u_x} + \zeta_1^2\frac{\partial}{\partial c_t} + \zeta_2^2\frac{\partial}{\partial c_x} + \zeta_{22}^2\frac{\partial}{\partial c_{xx}} \\ & + \omega_1^1\frac{\partial}{\partial f_t} + \omega_2^1\frac{\partial}{\partial f_x} + \omega_4^1\frac{\partial}{\partial f_c} + \omega_1^2\frac{\partial}{\partial g_t} + \omega_2^2\frac{\partial}{\partial g_x}\,\cdot \end{aligned} \tag{2.3}$$

Here the additional coordinates ζ and ω are determined by the prolongation formulae

$$\begin{aligned}
\zeta_1^1 &= D_t(\eta^1) - u_t D_t(\xi^1) - u_x D_t(\xi^2),\\
\zeta_2^1 &= D_x(\eta^1) - u_t D_x(\xi^1) - u_x D_x(\xi^2),\\
\zeta_1^2 &= D_t(\eta^2) - c_t D_t(\xi^1) - c_x D_t(\xi^2),\\
\zeta_2^2 &= D_x(\eta^2) - c_t D_x(\xi^1) - c_x D_x(\xi^2),
\end{aligned}$$

$$\begin{aligned}
\zeta_{22}^2 = \quad & u_{xx}\eta_u^2 + c_{xx}\eta_c^2 + (u_x)^2\eta_{uu}^2 + 2u_x c_x \eta_{uc}^2 + (c_x)^2\eta_{cc}^2\\
& -2c_{tx}D_x(\xi^1) - 2c_{xx}D_x(\xi^2) - c_t D_x^2(\xi^1) - c_x D_x^2(\xi^2)
\end{aligned}$$

and

$$\omega_1^1 = \widetilde{D}_t(\mu^1) - f_u\widetilde{D}_t(\eta^1),\ \omega_2^1 = \widetilde{D}_x(\mu^1) - f_u\widetilde{D}_x(\eta^1),\ \omega_4^1 = \widetilde{D}_c(\mu^1) - f_u\widetilde{D}_c(\eta^1),$$

$$\omega_1^2 = \widetilde{D}_t(\mu^2) - g_u\widetilde{D}_t(\eta^1) - g_c\widetilde{D}_t(\eta^2), \quad \omega_2^2 = \widetilde{D}_x(\mu^2) - g_u\widetilde{D}_x(\eta^1) - g_c\widetilde{D}_x(\eta^2),$$

respectively, where

$$D_t = \frac{\partial}{\partial t} + u_t\frac{\partial}{\partial u} + c_t\frac{\partial}{\partial c}\,, \quad D_x = \frac{\partial}{\partial x} + u_x\frac{\partial}{\partial u} + c_x\frac{\partial}{\partial c}$$

are the usual total differentiations, whereas

$$\widetilde{D}_t = \frac{\partial}{\partial t}\,, \quad \widetilde{D}_x = \frac{\partial}{\partial x}\,, \quad \widetilde{D}_c = \frac{\partial}{\partial c} + g_c\frac{\partial}{\partial g}$$

denote the "new" total differentiations for the extended system (2.2). The invariance test for the equations (2.1) is written as the following system of *determining equations:*

$$\zeta_1^1 - \mu^1 + c_x\zeta_2^1 + u_x\zeta_2^2 + \eta^1 c_{xx} + u\zeta_{22}^2 = 0, \qquad \zeta_1^2 + \mu^2 = 0, \tag{2.4}$$

$$\omega_1^1 = 0, \qquad \omega_2^1 = 0, \qquad \omega_4^1 = 0, \qquad \omega_1^2 = 0, \qquad \omega_2^2 = 0. \tag{2.5}$$

Let us begin with the equations (2.5). Using the first of them,

$$\omega_1^1 = \mu_t^1 - f_u\eta_t^1 = 0,$$

and invoking that $f(u)$ and hence $f_u = f'(u)$ are arbitrary functions, we obtaind

$$\mu_t^1 = 0, \qquad \eta_t^1 = 0.$$

Likewise the remaining equations (2.5) yield:

$$\omega_2^1 = \mu_x^1 - f_u\eta_x^1 = 0 \implies \mu_x^1 = 0,\ \eta_x^1 = 0,$$

$$\omega_4^1 = \mu_c^1 + g_c\mu_g^1 - f_u\eta_c^1 = 0 \implies \mu_c^1 = 0,\ \mu_g^1 = 0,\ \eta_c^1 = 0,$$
$$\omega_1^2 = \mu_t^2 - g_c\eta_t^2 = 0 \implies \mu_t^2 = 0,\ \eta_t^2 = 0,$$
$$\omega_2^2 = \mu_x^2 - g_c\eta_x^2 = 0 \implies \mu_x^2 = 0,\ \eta_x^2 = 0.$$

Thus,

$$\mu^1 = \mu^1(u, f), \quad \mu^2 = \mu^2(u, c, f, g), \quad \eta^1 = \eta^1(u), \quad \eta^2 = \eta^2(u, c). \tag{2.6}$$

The first equation in (2.4), taking into account (2.6), is written

$$D_t(\eta^1) - u_t D_t(\xi^1) - u_x D_t(\xi^2) + c_x \left[D_x(\eta^1) - u_t D_x(\xi^1) - u_x D_x(\xi^2)\right]$$
$$-\mu^1(u, f) + u_x \left[u_x\eta_u^2 + c_x\eta_c^2 - c_t D_x(\xi^1) - c_x D_x(\xi^2)\right] + \eta^1 c_{xx} + u\zeta_{22}^2 = 0.$$

Solving this equation and determining μ_2 from the second equation (2.4), we finally get the general solution of the determining equations (2.4)-(2.5) involving six arbitrary constants K :

$$\begin{gathered} \xi^1 = K_1 + 2K_3 t, \qquad \xi^2 = K_2 + (K_3 + K_5)x, \\ \eta^1 = K_6 u, \qquad \eta^2 = K_4 + 2K_5 c, \\ \mu^1 = (K_6 - 2K_3)f, \quad \mu^2 = 2(K_5 - K_3)g. \end{gathered} \tag{2.7}$$

Thus, the system (1.2) has the six-dimensional equivalence Lie algebra spanned by the following equivalence generators:

$$\begin{aligned} &Y_1 = \frac{\partial}{\partial t}, \qquad Y_2 = \frac{\partial}{\partial x}, \qquad Y_3 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} - 2f\frac{\partial}{\partial f} - 2g\frac{\partial}{\partial g}, \\ &Y_4 = \frac{\partial}{\partial c}, \qquad Y_5 = x\frac{\partial}{\partial x} + 2c\frac{\partial}{\partial c} + 2g\frac{\partial}{\partial g}, \qquad Y_6 = u\frac{\partial}{\partial u} + f\frac{\partial}{\partial f}. \end{aligned} \tag{2.8}$$

3 Applications of equivalence Lie algebra

3.1 Projections of equivalence generators

We will use the theorem on projections of equivalence generators (see [48], Paper 3) in a way similar to that in [66] (see also [67]).

Let us denote by $\boldsymbol{x} = (t, x)$ and $\boldsymbol{u} = (c, u)$ the *independent* and *dependent* variables, respectively, and by $\boldsymbol{f} = (f, g)$ the *arbitrary elements* in the system (1.2). Consider the projection

$$X = \mathrm{pr}_{(\boldsymbol{x},\boldsymbol{u})}(Y)$$

of the equivalence generator (2.1) to the space $(\boldsymbol{x}, \boldsymbol{u})$ of the independent and dependent variables, and the projection

$$Z = \mathrm{pr}_{(\boldsymbol{u},\boldsymbol{f})}(Y)$$

to the space $(\boldsymbol{u}, \boldsymbol{f})$ involved in the arbitrary elements:

$$X = \mathrm{pr}_{(\boldsymbol{x},\boldsymbol{u})}(Y) \equiv \xi^1 \frac{\partial}{\partial t} + \xi^2 \frac{\partial}{\partial x} + \eta^1 \frac{\partial}{\partial u} + \eta^2 \frac{\partial}{\partial c}\,, \tag{3.1}$$

$$Z = \mathrm{pr}_{(\boldsymbol{u},\boldsymbol{f})}(Y) \equiv \eta^1 \frac{\partial}{\partial u} + \eta^2 \frac{\partial}{\partial c} + \mu^1 \frac{\partial}{\partial f} + \mu^2 \frac{\partial}{\partial g}\,\cdot \tag{3.2}$$

The equations (2.7) manifest that the operators X and Z are well defined, i.e. their coordinates involve only the respective variables, $(\boldsymbol{x}, \boldsymbol{u})$ and $(\boldsymbol{u}, \boldsymbol{f})$. In our specific case the theorem on projections is formulated as follows.

Consider the equivalence generator

$$Y = K_1Y_1 + K_2Y_2 + K_3Y_3 + K_4Y_4 + K_5Y_5 + K_6Y_6$$

i.e. the linear combination of the operators (2.8):

$$Y = (K_1 + 2K_3t)\frac{\partial}{\partial t} + [K_2 + (K_3 + K_5)x]\frac{\partial}{\partial x} + K_6u\frac{\partial}{\partial u}$$

$$+(K_4 + 2K_5c)\frac{\partial}{\partial c} + (K_6 - 2K_3)f\frac{\partial}{\partial f} + 2(K_5 - K_3)g\frac{\partial}{\partial g}\,, \tag{3.3}$$

The projection (3.1) *of the operator* (3.3),

$$X = (K_1+2K_3t)\frac{\partial}{\partial t}+[K_2+(K_3+K_5)x]\frac{\partial}{\partial x}+K_6u\frac{\partial}{\partial u}+(K_4+2K_5c)\frac{\partial}{\partial c} \tag{3.4}$$

is admitted by equations (1.2) *with specific functions*

$$f = F(u), \quad g = G(c, u) \tag{3.5}$$

if and only if the constants K are chosen so that the projection (3.2),

$$Z = K_6u\frac{\partial}{\partial u} + (K_4 + 2K_5c)\frac{\partial}{\partial c} + (K_6 - 2K_3)f\frac{\partial}{\partial f} + 2(K_5 - K_3)g\frac{\partial}{\partial g} \tag{3.6}$$

is admitted by the equations (3.5). *In particular, $X \in L_{\mathcal{P}}$ if and only if the projection* (3.6) *vanishes:*

$$Z = \mathrm{pr}_{(\boldsymbol{u},\boldsymbol{f})}(Y) = 0. \tag{3.7}$$

3.2 Principal Lie algebra

The operator equation (3.7) provides a simple way to find the principal Lie algebra. Indeed, according to (3.6), equation (3.7) is written

$$[K_6 u\frac{\partial}{\partial u} + (K_4 + 2K_5 c)\frac{\partial}{\partial c} + (K_6 - 2K_3) f\frac{\partial}{\partial f} + 2(K_5 - K_3) g\frac{\partial}{\partial g} = 0,$$

whence

$$K_3 = 0, \quad K_4 = 0, \quad K_5 = 0, \quad K_6 = 0,$$

and hence

$$Y = K_1 Y_1 + K_2 Y_2.$$

Since

$$X_1 = \mathrm{pr}_{(\boldsymbol{x},\boldsymbol{u})}(Y_1) = Y_1, \quad X_2 = \mathrm{pr}_{(\boldsymbol{x},\boldsymbol{u})}(Y_2) = Y_2,$$

we conclude that the operators (1.3) span the principal Lie algebra $L_{\mathcal{P}}$.

3.3 A case with extended symmetry

Consider, as an example, the following equivalence generator:

$$Y = Y_4 + Y_6 \equiv \frac{\partial}{\partial c} + u\frac{\partial}{\partial u} + f\frac{\partial}{\partial f}\,\cdot \tag{3.8}$$

In this example, the operator Y coincides with its projection $\mathrm{pr}_{(\boldsymbol{u},\boldsymbol{f})}(Y)$:

$$Z = \mathrm{pr}_{(\boldsymbol{u},\boldsymbol{f})}(Y) = \frac{\partial}{\partial c} + u\frac{\partial}{\partial u} + f\frac{\partial}{\partial f}.$$

The invariance conditions for equations (3.5) are written

$$Z\,(f - F(u))\Big|_{f=F(u)} = F - uF' = 0,$$

$$Z\,(g - G(c,u))\Big|_{g=G(c,u)} = -\frac{\partial G}{\partial c} - u\frac{\partial G}{\partial u} = 0,$$

whence

$$f = \alpha u, \quad g = G(u\mathrm{e}^{-c}),$$

where α is an arbitrary constant and G an arbitrary function of one variable. The corresponding system (1.2),

$$u_t = \alpha u - (uc_x)_x, \quad c_t = -G(u\mathrm{e}^{-c})$$

admits, along with X_1, X_2 given in (1.3), the additional operator

$$X_3 = \text{pr}_{(\boldsymbol{x},\boldsymbol{u})}(Y) = \frac{\partial}{\partial c} + u\frac{\partial}{\partial u}\cdot$$

Letting, e.g.

$$G(u\mathrm{e}^{-c}) = u\mathrm{e}^{-c}$$

we consider the system

$$\begin{aligned} u_t &= \alpha u - (uc_x)_x, \\ c_t &= -u\mathrm{e}^{-c}. \end{aligned} \tag{3.9}$$

3.4 An invariant solution

Let us find invariant solutions using the operator

$$X_1 + X_3 = \frac{\partial}{\partial t} + \frac{\partial}{\partial c} + u\frac{\partial}{\partial u}\cdot$$

It has three independent invariants:

$$x, \quad \psi_1 = c - t, \quad \psi_2 = u\mathrm{e}^{-t}.$$

The corresponding invariant solutions have the form

$$c = t + \psi_1(x), \quad u = \mathrm{e}^t\psi_2(x). \tag{3.10}$$

It follows

$$u_t = \mathrm{e}^t\psi_2(x), \quad u_x = \mathrm{e}^t\psi_2'(x), \quad c_t = 1, \quad c_x = \psi_1'(x), \quad c_{xx} = \psi_1''(x). \tag{3.11}$$

Substituting the expressions (3.10) and (3.11) in the first equation (3.9), one obtains:

$$\mathrm{e}^t\psi_2(x) = \alpha\mathrm{e}^t\psi_2(x) - \mathrm{e}^t\psi_1'(x)\psi_2'(x) - \mathrm{e}^t\psi_2(x)\psi_1''(x),$$

or

$$(1-\alpha)\psi_2(x) + \psi_1'(x)\psi_2'(x) + \psi_2(x)\psi_1''(x) = 0. \tag{3.12}$$

The second equation (3.9) implies

$$1 = -\psi_2(x)\mathrm{e}^{-\psi_1(x)}$$

whence

$$\psi_2(x) = -\mathrm{e}^{\psi_1(x)}. \tag{3.13}$$

The equation (3.12), invoking (3.13), is rewritten

$$\psi_1''(x) + \psi_1'^2 + (1 - \alpha) = 0$$

and yields:

for $\alpha = 1$:

$$\psi_1(x) = \ln |\, A_2\,(x + A_1)|\,, \tag{3.14}$$

for $\alpha > 1$:

$$\psi_1(x) = x\sqrt{\alpha - 1} + \ln \left| A_2 \left(1 \pm \mathrm{e}^{2\sqrt{\alpha-1}\,(A_1 - x)}\right) \right|, \tag{3.15}$$

for $\alpha < 1$:

$$\psi_1(x) = \ln \left| A_2\, \cos\left(\sqrt{1-\alpha}\,(A_1 - x)\right) \right|, \tag{3.16}$$

where A_1 and A_2 are arbitrary constants.

Finally, substituting (3.14), (3.15) and (3.16) into the equations for c and u (3.10), invoking (3.13), one obtains three different solutions for the system (3.9):

$$\begin{aligned} c(t,x) &= t + \ln |\, A_2\,(x + A_1)|\,, \\ u(t,x) &= -\mathrm{e}^t\, |\, A_2\,(x + A_1)| \qquad (\alpha = 1); \end{aligned} \tag{3.17}$$

$$\begin{aligned} c(t,x) &= t + x\sqrt{\alpha-1} + \ln \left| A_2 \left(1 \pm \mathrm{e}^{2\sqrt{\alpha-1}\,(A_1 - x)}\right) \right|, \\ u(t,x) &= -\mathrm{e}^{t + x\sqrt{\alpha-1}} \left| A_2 \left(1 \pm \mathrm{e}^{2\sqrt{\alpha-1}\,(A_1 - x)}\right) \right| \quad (\alpha > 1); \end{aligned} \tag{3.18}$$

and

$$\begin{aligned} c(t,x) &= t + \ln \left| A_2 \cos\left(\sqrt{1-\alpha}\,(A_1 - x)\right) \right|, \\ u(t,x) &= -\mathrm{e}^t \left| A_2 \cos\left(\sqrt{1-\alpha}\,(A_1 - x)\right) \right| \quad (\alpha < 1), \end{aligned} \tag{3.19}$$

respectively. The solution (3.19) with $\alpha < 0$ is relevant to the model (3.9). Namely, the functions

$$f(u) = \alpha u, \quad g(c,u) = u\mathrm{e}^{-c}$$

satisfy the conditions for the model (1.2):

$$f(u) = \alpha u > 0, \quad g_c(c,u) = -u\mathrm{e}^{-c} > 0, \quad g_u(c,u) = \mathrm{e}^{-c} > 0.$$

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Paper 3

Invariants for generalised Burgers equations

Talk at MOGRAN-9
Moscow, 2002
(See also [63])

Abstract. Using the group of equivalence transformations and the infinitesimal method, we calculate invariants of a family of generalised Burgers equations which has applications in acoustic phenomena and furthermore has been used to model turbulence and certain steady state viscous flows.

1 Introduction

Here, we calculate invariants of the family of generalised Burgers equations

$$u_t + uu_x + f(t)u_{xx} = 0, \tag{1.1}$$

where $f(t)$ is an arbitrary function. The calculations are based on the infinitesimal method for computing invariants of families of differential equations by using equivalence groups (see [38]). The method was employed first for understanding the group theoretic nature of the well-known *Laplace invariants* for the linear hyperbolic partial differential equation with two independent varibles and then to derive the Laplace type invariants for the parabolic equations [41]. The method was also applied to families of non-linear equations.

If $f(t)$ is a constant then Equation (1.1) becomes the well known Burgers equation. The Burgers equation, among other applications, is used to model the formation and decay of non-planar shock waves, where the variable x

is a coordinate moving with the wave at the speed of the sound, and the dependent variable u represents the velocity fluctuations. The coefficient of u_{xx} in the equation is approximated by a constant, but is actually a function of the time, and hence there is merit in studying Equation (1.1) (see the survey [81] and the recent paper [18]).

The equivalence group of the family of equations (1.1) with an arbitrary function $f(t)$ is a certain representation in the (t, x, u) space of the projective group on the (t, x) plane. We show that the invariant of Equations (5.1) is a third-order differential invariant of the projective group and that this invariant is the *Schwarzian* which has remarkable properties. In the last section we consider a more general class of Burgers type equations.

2 Equivalence group

The main goal of the present paper is to derive the invariants for Equation (1.1) using the group of equivalence transformations found in [71].

2.1 Equivalence transformations

The equivalence group of Equation (1.1) contains the linear transformation

$$\begin{aligned} \bar{x} &= c_3c_5x + c_1c_5^2t + c_2, \\ \bar{t} &= c_5^2t + c_4, \\ \bar{u} &= \frac{c_3}{c_5}u + c_1 \end{aligned} \tag{2.1}$$

and the projective transformation

$$\begin{aligned} \bar{x} &= \frac{c_3c_6x - c_1}{c_6^2t - c_4} + c_2, \\ \bar{t} &= c_5 - \frac{1}{c_6^2t - c_4}, \\ \bar{u} &= c_3c_6(ut - x) + \frac{c_3c_4}{c_6}u + c_1, \end{aligned} \tag{2.2}$$

where $c_1, \ldots,\ c_6$ are constants such that

$$c_3 \neq 0, \quad c_5 \neq 0, \quad c_6 \neq 0.$$

Under both transformations (2.1) and (2.2) the coefficient $f(t)$ of Equation (1.1) is mapped to

$$\bar{f}(\bar{t}) = c_3^2 f(t). \tag{2.3}$$

2.2 Generators of the equivalence group

Denoting the infinitesimal generators of the equivalence group by

$$Y = T(t,x,u)\frac{\partial}{\partial t} + X(t,x,u)\frac{\partial}{\partial x} + U(t,x,u)\frac{\partial}{\partial u} + \eta(t,x,u,f)\frac{\partial}{\partial f} \tag{2.4}$$

one obtains from (2.2) and (2.3) the following six basic generators:

$$\begin{gathered} Y_1 = \frac{\partial}{\partial t}\,,\ Y_2 = \frac{\partial}{\partial x}\,,\ Y_3 = t\frac{\partial}{\partial x} + \frac{\partial}{\partial u}\,,\ Y_4 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} - u\frac{\partial}{\partial u}\,, \\ Y_5 = x\frac{\partial}{\partial x} + u\frac{\partial}{\partial u} + 2f\frac{\partial}{\partial f}\,,\ Y_6 = t^2\frac{\partial}{\partial t} + xt\frac{\partial}{\partial x} + (x-ut)\frac{\partial}{\partial u}\cdot \end{gathered} \tag{2.5}$$

3 Invariants

3.1 Calculation of invariants

Invariants $J(t,x,u,f)$ of the equivalence group are determined by equations

$$Y_k J = 0,\ \ k = 1,\dots,6. \tag{3.6}$$

Substituting here the operators (2.5), one readily obtains that

$$J_t = J_x = J_u = J_f = 0.$$

Thus, there are no invariants other than $J =$ constant the latter being a trivial invariant.

Hence, we have to consider first-order *differential invariants*, i.e. invariants of the form

$$J(t,x,y,f,f'), \quad f = \frac{df}{dt}\cdot$$

These are defined by the equations

$$Y_k^{(1)} J = 0,$$

where $Y_k^{(1)}$ denote the prolongation of Y_k to f'. These equations also lead to the trivial invariant $J =$ const. Likewise, using the second prolongation of the generators (2.5) and solving the corresponding equations

$$Y_k^{(2)} J = 0$$

we conclude that the second-order differential invariants

$$J(t,x,u,f,f',f'')$$

are are also trivial.

Therefore we search for third-order differential invariants

$$J(t,x,u,f,f',f'',f''')$$

which are given by

$$Y_k^{(3)}J=0,\quad k=1,\ldots,6, \tag{3.7}$$

where

$$\begin{aligned}
&Y_1^{(3)}=\frac{\partial}{\partial t},\ Y_2^{(3)}=\frac{\partial}{\partial x},\ Y_3^{(3)}=t\frac{\partial}{\partial x}+\frac{\partial}{\partial u},\\
&Y_4^{(3)}=2t\frac{\partial}{\partial t}+x\frac{\partial}{\partial x}-u\frac{\partial}{\partial u}-2f'\frac{\partial}{\partial f'}-4f''\frac{\partial}{\partial f''}-6f'''\frac{\partial}{\partial f'''},\\
&Y_5^{(3)}=x\frac{\partial}{\partial x}+u\frac{\partial}{\partial u}+2f\frac{\partial}{\partial f}+2f'\frac{\partial}{\partial f'}+2f''\frac{\partial}{\partial f''}+2f'''\frac{\partial}{\partial f'''}\\
&Y_6^{(3)}=t^2\frac{\partial}{\partial t}+xt\frac{\partial}{\partial x}+(x-ut)\frac{\partial}{\partial u}-2tf'\frac{\partial}{\partial f'}\\
&-2(2tf''+f')\frac{\partial}{\partial f''}-6(tf'''+f'')\frac{\partial}{\partial f'''}
\end{aligned} \tag{3.8}$$

The first three equations of (3.7) give

$$J_t=J_x=J_u=0.$$

Hence, the invariant is of the form $J(f,f',f'',f''')$ and the remaining equations in (3.7) become

$$\begin{aligned}
&f'\frac{\partial J}{\partial f'}+2f''\frac{\partial J}{\partial f''}+3f'''\frac{\partial J}{\partial f'''}=0,\\
&f\frac{\partial J}{\partial f}+f'\frac{\partial J}{\partial f'}+f''\frac{\partial J}{\partial f''}+f'''\frac{\partial J}{\partial f'''}=0,\\
&tf'\frac{\partial J}{\partial f'}+(2tf''+f')\frac{\partial J}{\partial f''}+3(tf'''+f'')\frac{\partial J}{\partial f''}=0.
\end{aligned} \tag{3.9}$$

Equations (3.9) provide the solution

$$J=\frac{f^2}{f'^3}\left(f'''-\frac{3}{2}\frac{f''^2}{f'}\right). \tag{3.10}$$

The expression in the brackets is known as the *Schwarzian*. It has remarkable properties due to its invariance under the projective group (see, e.g. [39], Section 1.2.3, Exercise 4).

3.2 Invariant differentiation

One can construct a sequence of higher-order differential invariants using the invariant differentiation (with respect to t),

$$\mathcal{D} = \lambda D_t,$$

associated with the generators (2.5). Here

$$D_t = \frac{\partial}{\partial t} + f'\frac{\partial}{\partial f} + f''\frac{\partial}{\partial f'} + f'''\frac{\partial}{\partial f''} + \cdots$$

is the total differentiation and the coefficient

$$\lambda = \lambda(f, f', f'', f''')$$

is defined by the following equations (see, e.g. [39], Section 8.3.5):

$$Y_k^{(3)}\lambda = \lambda D_t(T), \quad k = 1, \ldots, 6. \tag{3.11}$$

The solution of Equations (3.11) is

$$\lambda = \frac{f}{f'}\phi\left[\frac{f^2}{f'^4}\left(ff''' - \frac{3}{2}f''^2\right)\right]$$

with an arbitrary function ϕ. We set $\phi = 1$ and therefore

$$\lambda = \frac{f}{f'},$$

and obtain the sequence of invariants $J_n = \lambda D_t J_{n-1}$, i.e.

$$J_n = \frac{f}{f'}D_t J_{n-1}, \quad n = 1, 2, \ldots,$$

where $J_0 = J$ given by Equation (3.10).

4 Extension to equation $u_t + uu_x + f(x,t)u_{xx} = 0$

One can verify that (2.1) and (2.2) provide equivalence transformations for the generalised Burgers equation of the form

$$u_t + uu_x + f(x,t)u_{xx} = 0.$$

It turns out that non-trivial invariants exist when we apply the second prolongation of the generators (2.5) which are read as follows:

$$
\begin{aligned}
&Y_1^{(2)} = \frac{\partial}{\partial t},\ Y_2^{(2)} = \frac{\partial}{\partial x},\ Y_3^{(2)} = Y_3 - f_x\frac{\partial}{\partial f_t} - 2f_{xt}\frac{\partial}{\partial f_{tt}} - f_{xx}\frac{\partial}{\partial f_{tx}}\\
&Y_4^{(2)} = Y_4 - 2f_t\frac{\partial}{\partial f_t} - f_x\frac{\partial}{\partial f_x} - 4f_{tt}\frac{\partial}{\partial f_{tt}} - 3f_{tx}\frac{\partial}{\partial f_{tx}} - 2f_{xx}\frac{\partial}{\partial f_{xx}}\\
&Y_5^{(2)} = Y_5 - 2f_t\frac{\partial}{\partial f_t} - 3f_x\frac{\partial}{\partial f_x} - 2f_{tt}\frac{\partial}{\partial f_{tt}} - 3f_{tx}\frac{\partial}{\partial f_{tx}} - 2f_{xx}\frac{\partial}{\partial f_{xx}}\\
&Y_6^{(2)} = Y_6 - (2tf_t + xf_x)\frac{\partial}{\partial f_t} - tf_x\frac{\partial}{\partial f_x} -\\
&(2f_t + 4tf_{tt} + 2xf_{tx})\frac{\partial}{\partial f_{tt}} - (3tf_{tx} + xf_{xx} + f_x)\frac{\partial}{\partial f_{tx}} - 2tf_{xx}\frac{\partial}{\partial f_{xx}}
\end{aligned}
$$

The solution of the six equations

$$Y_k^{(2)} J = 0, \quad k = 1, \ldots,\ 6,$$

provides the following two invariants:

$$J_1 = \frac{f_x^2}{ff_{xx}}, \quad J_2 = \frac{f^2}{f_{xx}^3}(2f_tf_xf_{tx} - f_t^2 f_{xx} - f_x^2 f_{tt}).$$

Paper 4

Alternative proof of Lie's linearization theorem

Preprint of the paper [56]

Abstract. S. Lie found the general form of all second-order ordinary differential equations transformable to the linear equation by a change of variables. He showed that the linearizable equations are at most cubic in the first-order derivative and described a procedure for constructing linearizing transformations. Using a projective geometric reasoning, Lie found an over-determined system of four auxiliary equations for two variables and proved that the compatibility of these auxiliary equations furnishes a necessary and sufficient condition for linearization. This result is known as Lie's linearization theorem. We present here an alternative proof of Lie's theorem using techniques of Riemannian geometry.

1 Introduction

S. Lie, in his general theory of integration of ordinary differential equations admitting a group of transformations, proved *inter alia* ([77], §1) that if a non-linear equation of second order $y'' = f(x, y, y')$ is transformable to a linear equation by a change of variables x, y, its integration requires only quadratures and solution of a linear third-order ordinary differential equation.

As a first step, Lie showed that the linearizable second-order equations are at most cubic in the first derivative, i.e. belong to the family of equations of the form

$$y'' + F_3(x, y)y'^{\,3} + F_2(x, y)y'^{\,2} + F_1(x, y)y' + F(x, y) = 0. \tag{1.1}$$

Furthermore, he found that the following over-determined system of four equations is compatible for linearizable equations (see [77], §1, equations (3)-(4)):

$$\begin{aligned}
\frac{\partial w}{\partial x} &= zw - FF_3 - \frac{1}{3}\frac{\partial F_1}{\partial y} + \frac{2}{3}\frac{\partial F_2}{\partial x},\\
\frac{\partial w}{\partial y} &= -w^2 + F_2 w + F_3 z + \frac{\partial F_3}{\partial x} - F_1 F_3,\\
\frac{\partial z}{\partial x} &= z^2 - Fw - F_1 z + \frac{\partial F}{\partial y} + FF_2,\\
\frac{\partial z}{\partial y} &= -zw + FF_3 - \frac{1}{3}\frac{\partial F_2}{\partial x} + \frac{2}{3}\frac{\partial F_1}{\partial y},
\end{aligned} \tag{1.2}$$

and used this system as a basis for construction of linearizing transformations. One can verify that the compatibility conditions of the system (1.2) are provided by the following equations:

$$3(F_3)_{xx} - 2(F_2)_{xy} + (F_1)_{yy} = 3(F_1F_3)_x - 3(FF_3)_y - (F_2^2)_x - 3F_3F_y + F_2(F_1)_y,$$

$$3F_{yy} - 2(F_1)_{xy} + (F_2)_{xx} = 3(FF_3)_x - 3(FF_2)_y + (F_1^2)_y + 3F(F_3)_x - F_1(F_2)_x,$$

where the subscripts x and y denote differentiations in x and y, respectively.

Finally, he demonstrated in [77], Note 1 (see also p. 423 in [80], vol. 5) the *linearization test* stating that Equation (1.1) is linearizable if and only if the over-determined system (1.2) is compatible. Lie's linearization test is indeed simple and convenient in practice. Consider the following examples (see also [39], Section 12.3).

Example 1. The equation

$$y'' + F(x, y) = 0$$

has the form (1.1) with $F_3 = F_2 = F_1 = 0$. The compatibility of the system (1.2) requires that $F_{yy} = 0$. Hence, the equation $y'' + F(x, y) = 0$ cannot be linearized unless it is already linear.

Example 2. The equations

$$y'' - \frac{1}{x}(y' + y'^3) = 0$$

and

$$y'' + \frac{1}{x}(y' + y'^3) = 0$$

also have the form (1.1). Their coefficients are $F_3 = F_1 = -1/x, F_2 = F = 0$ and $F_3 = F_1 = 1/x, F_2 = F = 0$, respectively. The linearization test shows that the first equation is linearizable, whereas the second one is not.

2 Outline of Lie's approach

2.1 Derivation of Equation (1.1)

Recall that any linear equation of the second order

$$y'' + a(x)y' + b(x)y = 0$$

can be reduced, by a change of variables, to the simplest form

$$\frac{d^2u}{dt^2} = 0. \tag{2.3}$$

Therefore all linearizable equations

$$y'' = f(x, y, y')$$

are obtained from Equation (2.3) by an arbitrary change of variables

$$t = \phi(x, y), \quad u = \psi(x, y). \tag{2.4}$$

In the new variables x and y defined by (2.4), Equation (2.3) is written

$$y'' + A\,y'^{\,3} + (B + 2w)\,y'^{\,2} + (P + 2z)\,y' + Q = 0, \tag{2.5}$$

where (see, e.g. [39], Section 12.3)

$$\begin{aligned}
A &= \frac{\phi_y\psi_{yy} - \psi_y\phi_{yy}}{\phi_x\psi_y - \phi_y\psi_x}\,, \quad B = \frac{\phi_x\psi_{yy} - \psi_x\phi_{yy}}{\phi_x\psi_y - \phi_y\psi_x}\,,\\
P &= \frac{\phi_y\psi_{xx} - \psi_y\phi_{xx}}{\phi_x\psi_y - \phi_y\psi_x}\,, \quad Q = \frac{\phi_x\psi_{xx} - \psi_x\phi_{xx}}{\phi_x\psi_y - \phi_y\psi_x}\,,\\
w &= \frac{\phi_y\psi_{xy} - \psi_y\phi_{xy}}{\phi_x\psi_y - \phi_y\psi_x}\,, \quad z = \frac{\phi_x\psi_{xy} - \psi_x\phi_{xy}}{\phi_x\psi_y - \phi_y\psi_x}\,.
\end{aligned} \tag{2.6}$$

Equation (2.5) takes the form (1.1) upon denoting

$$\begin{aligned}
&A = F_3(x, y), \quad B + 2w = F_2(x, y),\\
&P + 2z = F_1(x, y), \quad Q = F(x, y).
\end{aligned} \tag{2.7}$$

Thus, any linearizable equation has the form (1.1) with the coefficients $F_3(x, y)$, $F_2(x, y)$, $F_1(x, y)$, $F_1(x, y)$ defined by (2.7) and (2.6).

Consider now an arbitrary equation of the form (1.1). According to the above calculations, it is linearizable if and only if the equations (2.7) hold, i.e. if the system

$$
\begin{aligned}
&\phi_y\psi_{yy} - \psi_y\phi_{yy} = F_3(x,y)\,(\phi_x\psi_y - \phi_y\psi_x),\\
&\phi_x\psi_{yy} - \psi_x\phi_{yy} + 2(\phi_y\psi_{xy} - \psi_y\phi_{xy}) = F_2(x,y)\,(\phi_x\psi_y - \phi_y\psi_x),\\
&\phi_y\psi_{xx} - \psi_y\phi_{xx} + 2(\phi_x\psi_{xy} - \psi_x\phi_{xy}) = F_1(x,y)\,(\phi_x\psi_y - \phi_y\psi_x),\\
&\phi_x\psi_{xx} - \psi_x\phi_{xx} = F(x,y)\,(\phi_x\psi_y - \phi_y\psi_x),
\end{aligned}
\tag{2.8}
$$

with given coefficients $F_3(x,y),\ldots,F(x,y)$ and two unknown functions, ϕ and ψ, is compatible. We summarize.

Proposition 4.1. Equation (1.1) is linearizable if and only if its coefficients $F_3(x,y)$, $F_2(x,y)$, $F_1(x,y)$ and $F(x,y)$ are such that the over-determined system (2.8) is compatible. Provided that the system (2.8) is compatible, its integration furnishes a transformation (2.4) of the corresponding equation (1.1) to the linear equation (2.3).

2.2 Necessity of compatibility of the system (1.2)

Thus, one has primarily to find compatibility conditions for the over-determined system of nonlinear equations (2.8). Lie's crucial observation is that the combinations

$$A, \quad B + 2w, \quad P + 2z, \quad Q$$

of the quantities (2.6) are differential invariants of the general projective transformation

$$\overline{\phi} = \frac{L_2\,\psi + M_2\,\phi + N_2}{L\,\psi + M\,\phi + N}\,, \quad \overline{\psi} = \frac{L_1\,\psi + M_1\,\phi + N_1}{L\,\psi + M\,\phi + N} \tag{2.9}$$

of ϕ and ψ, where $L, M, \ldots, M_2, N_2$ are arbitrary constants. Using this observation, he found four relations connecting the six quantities (2.6) and their first-order derivatives with respect to x and y. Then, eliminating A, B, P, Q by means of equations (2.7), he arrived at the equations (1.2) thus proving that compatibility of equations (1.2) is necessary for equation (1.1) to be linearizable.

2.3 Sufficiency of compatibility of the system (1.2)

To prove that compatibility of equations (1.2) is sufficient for linearization, Lie used the following reasoning. The projective invariance hints that equations (1.2) can be linearized by introducing the homogeneous projective coordinates

$$w = \frac{\widetilde{w}}{v}, \quad z = \frac{\widetilde{z}}{v} \cdot$$

Furthermore, Lie noticed that the resulting linear system belongs to a special type of linear systems that can be reduced, by his general theory, to a linear third-order ordinary differential equation, provided that the coefficients F_3, F_2, F_1, F of equation (1.1) satisfy the compatibility conditions of the system (1.2). Thus, the quantities w and z can be found by solving a linear third-order ordinary differential equation.

Lie's further observation is that equations (1.2) hold when one replaces B, w, Q, z by

$$B_1 = \frac{\phi_x\psi_{yy} - \psi_x\phi_{yy}}{\phi_x\psi_y - \phi_y\psi_x} - 2\frac{\phi_y}{\phi}, \quad w_1 = \frac{\phi_y\psi_{xy} - \psi_y\phi_{xy}}{\phi_x\psi_y - \phi_y\psi_x} + \frac{\phi_y}{\phi},$$
$$Q_1 = \frac{\phi_x\psi_{xx} - \psi_x\phi_{xx}}{\phi_x\psi_y - \phi_y\psi_x} + 2\frac{\phi_x}{\phi}, \quad z_1 = \frac{\phi_x\psi_{xy} - \psi_x\phi_{xy}}{\phi_x\psi_y - \phi_y\psi_x} - \frac{\phi_x}{\phi}$$

as well as by

$$B_2 = \frac{\phi_x\psi_{yy} - \psi_x\phi_{yy}}{\phi_x\psi_y - \phi_y\psi_x} - 2\frac{\psi_y}{\psi}, \quad w_2 = \frac{\phi_y\psi_{xy} - \psi_y\phi_{xy}}{\phi_x\psi_y - \phi_y\psi_x} + \frac{\psi_y}{\psi},$$
$$Q_2 = \frac{\phi_x\psi_{xx} - \psi_x\phi_{xx}}{\phi_x\psi_y - \phi_y\psi_x} + 2\frac{\psi_x}{\psi}, \quad z_2 = \frac{\phi_x\psi_{xy} - \psi_x\phi_{xy}}{\phi_x\psi_y - \phi_y\psi_x} - \frac{\psi_x}{\psi} \cdot$$

Therefore, one can find the quantities w_1 and z_1, as well as w_2 and z_2, by solving the previous linear third-order ordinary differential equation. In consequence, one obtains

$$\frac{\phi_x}{\phi} = z - z_1, \quad \frac{\phi_y}{\phi} = w_1 - w,$$

$$\frac{\psi_x}{\phi} = z - z_2, \quad \frac{\psi_y}{\phi} = w_2 - w.$$

The quadrature provides the solution of the nonlinear system (2.8), and hence completes the determination of a linearizing transformation (2.4).

3 Alternative approach

We formulate now Lie's linearization test as follows and provide its alternative proof.

Theorem 4.1. A necessary and sufficient condition that the equation

$$y'' + F_3(x,y)y'^{\,3} + F_2(x,y)y'^{\,2} + F_1(x,y)y' + F(x,y) = 0 \tag{3.10}$$

be linearizable is that its coefficients F_3, F_2, F_1, F satisfy the equations

$$\begin{aligned} 3(F_3)_{xx} - 2(F_2)_{xy} + (F_1)_{yy} &= 3(F_1F_3)_x - 3(FF_3)_y \\ &\quad - (F_2^2)_x - 3F_3F_y + F_2(F_1)_y, \\ 3F_{yy} - 2(F_1)_{xy} + (F_2)_{xx} &= 3(FF_3)_x - 3(FF_2)_y \\ &\quad + (F_1^2)_y + 3F(F_3)_x - F_1(F_2)_x. \end{aligned} \tag{3.11}$$

Proof. *Necessity.* Equations (3.11) provide necessary conditions for linearizable equations (3.10). Indeed, Equations (3.11) are satisfied for the linear equation $y'' = 0$. Furthermore, the changes of variables (2.4) are equivalence transformations for the set of all equations of the form (3.10). In other words, the equations (3.10) are merely permuted among themselves by any change of variables (2.4). It can be shown (see, e.g. [40]) that the system of equations (3.11) is invariant under the equivalence transformations (2.4). It follows from the invariance that Equations (3.11) hold for all equations (3.10) obtained from $y'' = 0$ by the changes of variables (2.4).

Sufficiency. Let us prove that Equations (3.11) provide sufficient conditions for Equation (3.10) to be linearizable. We consider plane curves given in a parametric form:

$$x = x(t), \quad y = y(t), \tag{3.12}$$

set $y(t) = u(x(t)),\ u' = du/dx$ and represent Equation (3.10) in the form

$$u'' + F_3u'^{\,3} + F_2u'^{\,2} + F_1u' + F = 0. \tag{3.13}$$

Then, denoting

$$\dot{x} = \frac{dx}{dt}\,, \quad \dot{y} = \frac{dy}{dt}\,, \tag{3.14}$$

we have:

$$\dot{y} = u'\,\dot{x}, \quad \ddot{y} = u''\,\dot{x}^2 + u'\,\ddot{x}, \quad \dot{x}^3u'' = \dot{x}\,\ddot{y} - \dot{y}\,\ddot{x},$$

and

$$\begin{aligned} &\dot{x}^3\left[u'' + F_3(x,y)u'^{\,3} + F_2(x,y)u'^{\,2} + F_1(x,y)u' + F(x,y)\right] \\ &= \dot{x}\left[\ddot{y} + \alpha\dot{y}^2 + \gamma\dot{x}\dot{y} + F\dot{x}^2\right] - \dot{y}\left[\ddot{x} - (F_3\,\dot{y}^2 + \beta\dot{x}\dot{y} + \delta\dot{x}^2)\right], \end{aligned}$$

where

$$\alpha + \beta = F_2, \quad \gamma + \delta = F_1. \tag{3.15}$$

Hence, Equation (3.13) becomes

$$\dot{x}\left[\ddot{y}+\alpha\dot{y}^2+\gamma\dot{x}\dot{y}+F\dot{x}^2\right]-\dot{y}\left[\ddot{x}-(F_3\,\dot{y}^2+\beta\dot{x}\dot{y}+\delta\dot{x}^2)\right]=0. \tag{3.16}$$

Upon rewriting Equation (3.10), or (3.13) in the form (3.16), we consider its projection onto the (x, y) plane. Namely, we split Equation (3.16) into the system

$$\begin{aligned}&\ddot{x}-F_3\,\dot{y}^2-\beta\dot{x}\dot{y}-\delta\dot{x}^2=0,\\&\ddot{y}+\alpha\dot{y}^2+\gamma\dot{x}\dot{y}+F\dot{x}^2=0\end{aligned} \tag{3.17}$$

and obtain the geodesic flow:

$$\ddot{x}^i+\Gamma^i_{kl}\dot{x}^k\,\dot{x}^l=0,\quad i=1,2, \tag{3.18}$$

where $x^1 = x, x^2 = y$. Comparison of Equations (3.17) and (3.18) shows that the Christoffel symbols have the form

$$\begin{aligned}&\Gamma^1_{11}=-\delta,\quad \Gamma^1_{12}=\Gamma^1_{21}=-\frac{1}{2}\beta,\quad \Gamma^1_{22}=-F_3,\\&\Gamma^2_{11}=F,\quad \Gamma^2_{12}\ =\ \Gamma^2_{21}\ =\ \frac{1}{2}\gamma,\quad \Gamma^2_{22}=\alpha.\end{aligned} \tag{3.19}$$

Equation (2.3) describes the straight lines on the (x, y) plane. Hence, to prove the theorem, it suffices to show that Equations (3.11) guarantee that the curves (3.12) can be straighten out. In other words, we have to show that if Equations (3.11) are satisfied, we can annul the Christoffel symbols (3.19) by a change of the variables x and y. It is possible if the Riemann tensor

$$R^l_{ijk}=\frac{\partial\Gamma^l_{ik}}{\partial x^j}-\frac{\partial\Gamma^l_{ij}}{\partial x^k}+\Gamma^m_{ik}\Gamma^l_{mj}-\Gamma^m_{ij}\Gamma^l_{mk} \tag{3.20}$$

associated with the Christoffel symbols (3.19) vanishes.

It is manifest from (3.20) that

$$R^l_{ijk}=-R^l_{ikj}.$$

Therefore we calculate only

$$R^l_{112}=0,\quad R^l_{212}=0,\quad l=1,2,$$

and obtain:

$$
\begin{aligned}
R^1_{112} &= \frac{\partial \Gamma^1_{12}}{\partial x} - \frac{\partial \Gamma^1_{11}}{\partial y} + \Gamma^m_{12}\Gamma^1_{m1} - \Gamma^m_{11}\Gamma^1_{m2}\\
&= \frac{\partial \Gamma^1_{12}}{\partial x} - \frac{\partial \Gamma^1_{11}}{\partial y} + \Gamma^2_{12}\Gamma^1_{21} - \Gamma^2_{11}\Gamma^1_{22}\\
&= -\frac{1}{2}\,\beta_x + \delta_y - \frac{1}{4}\,\beta\,\gamma + FF_3,\\
R^1_{212} &= \frac{\partial \Gamma^1_{22}}{\partial x} - \frac{\partial \Gamma^1_{21}}{\partial y} + \Gamma^m_{22}\Gamma^1_{m1} - \Gamma^m_{21}\Gamma^1_{m2}\\
&= \frac{\partial \Gamma^1_{22}}{\partial x} - \frac{\partial \Gamma^1_{21}}{\partial y} + \Gamma^1_{22}\Gamma^1_{11} + \Gamma^2_{22}\Gamma^1_{21} - \Gamma^1_{21}\Gamma^1_{12} - \Gamma^2_{21}\Gamma^1_{22}\\
&= -(F_3)_x + \frac{1}{2}\,\beta_y + \delta F_3 - \frac{1}{2}\,\alpha\,\beta - \frac{1}{4}\,\beta^2 + \frac{1}{2}\,\gamma F_3,
\end{aligned}
$$

$$
\begin{aligned}
R^2_{112} &= \frac{\partial \Gamma^2_{12}}{\partial x} - \frac{\partial \Gamma^2_{11}}{\partial y} + \Gamma^m_{12}\Gamma^2_{m1} - \Gamma^m_{11}\Gamma^2_{m2}\\
&= \frac{\partial \Gamma^2_{12}}{\partial x} - \frac{\partial \Gamma^2_{11}}{\partial y} + \Gamma^1_{12}\Gamma^2_{11} + \Gamma^2_{12}\Gamma^2_{21} - \Gamma^1_{11}\Gamma^2_{12} - \Gamma^2_{11}\Gamma^2_{22}\\
&= \frac{1}{2}\,\gamma_x - F_y - \frac{1}{2}\,\beta\,F + \frac{1}{4}\,\gamma^2 + \frac{1}{2}\,\gamma\,\delta - \alpha\,F,
\end{aligned}
$$

$$
\begin{aligned}
R^2_{212} &= \frac{\partial \Gamma^2_{22}}{\partial x} - \frac{\partial \Gamma^2_{21}}{\partial y} + \Gamma^m_{22}\Gamma^2_{m1} - \Gamma^m_{21}\Gamma^2_{m2}\\
&= \frac{\partial \Gamma^2_{22}}{\partial x} - \frac{\partial \Gamma^2_{21}}{\partial y} + \Gamma^1_{22}\Gamma^2_{11} - \Gamma^1_{21}\Gamma^2_{12}\\
&= \alpha_x - \frac{1}{2}\,\gamma_y - FF_3 + \frac{1}{4}\,\beta\,\gamma.
\end{aligned}
$$

Thus

$$\begin{aligned} R^1_{112} &= -\frac{1}{2}\,\beta_x + \delta_y - \frac{1}{4}\,\beta\,\gamma + F F_3,\\ R^1_{212} &= \frac{1}{2}\,\beta_y - (F_3)_x - \frac{1}{4}\,\beta^2 + \delta F_3 + \frac{1}{2}\,\gamma F_3 - \frac{1}{2}\,\alpha\,\beta,\\ R^2_{112} &= \frac{1}{2}\,\gamma_x - F_y + \frac{1}{4}\,\gamma^2 - \alpha\,F + \frac{1}{2}\,\gamma\,\delta - \frac{1}{2}\,\beta\,F,\\ R^2_{212} &= \alpha_x - \frac{1}{2}\,\gamma_y - F F_3 + \frac{1}{4}\,\beta\,\gamma. \end{aligned} \tag{3.21}$$

The equations

$$R^l_{ijk} = 0 \tag{3.22}$$

provide four first-order partial differential equations involving eight quantities F_3, F_2, F_1, F, α, β, γ and δ. Invoking that these quantities are connected by two conditions (3.15), we substitute

$$\alpha = F_2 - \beta, \quad \delta = F_1 - \gamma$$

in (3.21) and write Equations (3.22),

$$\begin{aligned} &-\frac{1}{2}\,\beta_x - \gamma_y - \frac{1}{4}\,\beta\,\gamma + F F_3 = 0,\\ &\frac{1}{2}\,\beta_y - (F_3)_x - \frac{1}{4}\,\beta^2 + \delta F_3 + \frac{1}{2}\,\gamma F_3 - \frac{1}{2}\,\alpha\,\beta = 0,\\ &\frac{1}{2}\,\gamma_x - F_y + \frac{1}{4}\,\gamma^2 - \alpha\,F + \frac{1}{2}\,\gamma\,\delta - \frac{1}{2}\,\beta\,F = 0,\\ &\alpha_x - \frac{1}{2}\,\gamma_y - F F_3 + \frac{1}{4}\,\beta\,\gamma = 0, \end{aligned}$$

as follows:

$$\begin{aligned} &-\frac{1}{2}\,\beta_x - \gamma_y + (F_1)_y - \frac{1}{4}\,\beta\,\gamma + F F_3 = 0,\\ &\frac{1}{2}\,\beta_y - (F_3)_x + \frac{1}{4}\,\beta^2 + F_1 F_3 - \frac{1}{2}\,\gamma F_3 - \frac{1}{2}\,\beta F_2 = 0,\\ &\frac{1}{2}\,\gamma_x - F_y - \frac{1}{4}\,\gamma^2 - F F_2 + \frac{1}{2}\,\gamma\,F_1 + \frac{1}{2}\,\beta\,F = 0,\\ &-\beta_x - \frac{1}{2}\,\gamma_y + (F_2)_x - F F_3 + \frac{1}{4}\,\beta\,\gamma = 0, \end{aligned} \tag{3.23}$$

We solve Equations (3.23) with respect to the derivatives of β and γ :

$$\frac{1}{2}\beta_x = \frac{1}{4}\beta\gamma - FF_3 - \frac{1}{3}(F_1)_y + \frac{2}{3}(F_2)_x, \tag{3.24}$$

$$\frac{1}{2}\beta_y = -\frac{1}{4}\beta^2 + \frac{1}{2}\beta F_2 + \frac{1}{2}\gamma F_3 + (F_3)_x - F_1F_3, \tag{3.25}$$

$$\frac{1}{2}\gamma_x = \frac{1}{4}\gamma^2 - \frac{1}{2}\beta F - \frac{1}{2}\gamma F_1 + F_y + FF_2, \tag{3.26}$$

$$\frac{1}{2}\gamma_y = -\frac{1}{4}\beta\gamma + FF_3 - \frac{1}{3}(F_2)_x + \frac{2}{3}(F_1)_y, \tag{3.27}$$

denote

$$w = \frac{\beta}{2}, \quad z = \frac{\gamma}{2}$$

and arrive at Lie's equations (1.2) compatibility of which is guaranteed by Equations (3.11). This completes the proof.

Remark 4.1. Another way to obtain (3.16) is to consider the first equation (3.12) as a change of the independent variable. Using the notation (3.14), we have:

$$y' = \frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{\dot{y}}{\dot{x}}\,,$$

$$y'' = \frac{dy'}{dt}\frac{dt}{dx} = \frac{d}{dt}\left(\frac{\dot{y}}{\dot{x}}\right)\frac{1}{\dot{x}} = \frac{1}{\dot{x}^3}\,(\dot{x}\ddot{y} - \dot{y}\ddot{x})\,.$$

Therefore

$$y'' + F_3\,y'^{\,3} + F_2\,y'^{\,2} + F_1\,y' + F$$

$$= \frac{1}{\dot{x}^3}\left(\dot{x}\ddot{y} - \dot{y}\ddot{x} + F_3\,\dot{y}^3 + F_2\,\dot{x}\dot{y}^2 + F_1\,\dot{x}^2\dot{y} + F\,\dot{x}^3\right)\,,$$

and Eq. (3.10) can be written in the form (3.16) with the notation (3.15).

Remark 4.2. The crucial moment of the alternative approach presented in Section 3 is that we transform the cubic polynomial in y' of Equation (3.10) into quadratic forms in $\dot{x}, \dot{y}$ of the system of equations (3.16). This allows us to tackle the linearization problem in terms of the Riemannian geometry.

Paper 5

Equivalence groups and invariants of linear and nonlinear equations

Revised survey [42]

Abstract. Recently I developed a systematic method for determining invariants of families of equations. The method is based on the infinitesimal approach and is applicable to algebraic and differential equations possessing finite or infinite equivalence groups. Moreover, it does not depend on the assumption of linearity of equations. The method was applied to variety of ordinary and partial differential equations. The present paper is aimed at discussing the main principles of the method and its applications with emphasis on the use of infinite Lie groups.

1 Introduction

The concept of invariants of differential equations is commonly in the case of linear second-order ordinary differential equations

$$y'' + 2c_1(x)y' + c_2(x)y = 0.$$

Namely, the linear substitution (an equivalence transformation)

$$\tilde{y} = \sigma(x)y$$

maps our equation again in a linear second-order equation and does not change the value of the the quantity

$$J = c_2 - c_1^2 - c_1'.$$

Knowledge of the invariant is useful in integration of differential equations. For instance, the equation

$$x^2 y'' + xy' + \left(x^2 - \frac{1}{4}\right) y = 0$$

has the invariant $J = 1$. The invariant $\widetilde{J}$ of the equation

$$\tilde{y}'' + \tilde{y} = 0$$

has the same value, $\widetilde{J} = 1$. In consequence, the first equation can be reduced to the second one by an equivalence transformation and hence readily integrated.

Mathematicians came across invariant quantities for families of equations in the very beginning of the theory of partial differential equations. The first partial differential equation, the wave equation $u_{xy} = 0$ for vibrating strings, was formulated and solved by d'Alembert in 1747. Two invariant quantities, h and k, for linear hyperbolic equations were found in 1769/1770 by Euler [19], then in 1773 by Laplace [75]. These fundamental invariant quantities are known today as the *Laplace invariants.*

We owe to Leonard Euler the first significant results in integration theory of general hyperbolic equations with two independent variables x, y :

$$u_{xy} + a(x,y)u_x + b(x,y)u_y + c(x,y)u = 0. \tag{1.1}$$

In his "Integral calculus" [19], Euler introduced what is known as the Laplace invariants, h and k. Namely, he generalized d'Alembert's solution and showed that Eq. (1.1) is factorable, and hence integrable by solving two first-order ordinary differential equations, if and only if its coefficients a, b, c obey one of the following equations:

$$h \equiv a_x + ab - c = 0, \tag{1.2}$$

or

$$k \equiv b_y + ab - c = 0. \tag{1.3}$$

If $h = 0$, Eq. (1.1) is factorable in the form

$$\left(\frac{\partial}{\partial x} + b\right)\left(\frac{\partial u}{\partial y} + au\right) = 0. \tag{1.4}$$

Then setting

$$v = u_y + au \tag{1.5}$$

we rewrite Eq. (1.4) as a first-order equation $v_x + bv = 0$ and integrate to obtain:

$$v = B(y)\mathrm{e}^{-\int b(x,y)dx}. \tag{1.6}$$

Now substitute (1.6) in (1.5), integrate the resulting non-homogeneous linear equation

$$u_y + au = B(y)\mathrm{e}^{-\int b(x,y)dx} \tag{1.7}$$

with respect to y and obtain the following general solution to Eq. (1.1):

$$u = \Big[A(x) + \int B(y)\mathrm{e}^{\int ady - bdx}dy\Big]\mathrm{e}^{-\int ady} \tag{1.8}$$

with two arbitrary functions $A(x)$ and $B(y)$.

Likewise, if $k = 0$, Eq. (1.1) is factorable in the form

$$\left(\frac{\partial}{\partial y} + a\right)\left(\frac{\partial u}{\partial x} + bu\right) = 0. \tag{1.9}$$

In 1773, Laplace [75] developed a more general method than that of Euler. In Laplace's method, known also as the *cascade method*, the quantities h, k play the central part. Laplace introduced two non-point equivalence transformations. Laplace's first transformation has the form

$$v = u_y + au, \tag{1.10}$$

and the second transformation has the form

$$w = u_x + bu. \tag{1.11}$$

Laplace's transformations allow one to solve some equations when both Laplace invariants are different from zero. Thus, let us we assume that $h \neq 0, k \neq 0$ and consider the the transformation (1.10). It maps Eq. (1.1) to the equation

$$v_{xy} + a_1 v_x + b_1 v_y + c_1 v = 0 \tag{1.12}$$

with the following coefficients:

$$a_1 = a - \frac{\partial \ln|h|}{\partial y}, \quad b_1 = b, \quad c_1 = c + b_y - a_x - b\frac{\partial \ln|h|}{\partial y}\cdot \tag{1.13}$$

The formulae (1.2) give the following Laplace invariants for Eq. (1.12):

$$h_1 = 2h - k - \frac{\partial^2 \ln|h|}{\partial x \partial y}, \quad k_1 = h. \tag{1.14}$$

Likewise, one can utilize the second transformation (1.11) and arrive to a linear equation for w with the Laplace invariants

$$h_2 = k, \quad k_2 = 2k - h - \frac{\partial^2 \ln |k|}{\partial x \partial y} \cdot \tag{1.15}$$

If $h_1 = 0$, one can solve Eq. (1.12) using Euler's method described above. Then it remains to substitute the solution $v = v(x, y)$ in (1.10) and to integrate the non-homogeneous first-order linear equation (1.10) for u. If $h_1 \neq 0$ but $k_2 = 0$, we find in a similar way the function $w = w(x, y)$ and solve the non-homogeneous first-order linear equation (1.11) for u. If $h_1 \neq 0$ and $k_2 \neq 0$, one can iterate the Laplace transformations by applying the transformations (1.10) and (1.11) to equations for v and w, etc. This is the essence of the *cascade method.*

In the 1890s, Darboux discovered the invariance of h and k and called them the *Laplace invariants.* He also simplified and improved Laplace's method, and the method became widely known due to Darboux's excellent presentation (see [17], Book IV, Chapters 2-9). Since the quantities h and k are invariant only under a subgroup of the equivalence group rather than the entire equivalence group, I proposed [38]* to call h and k the *semi-invariants* in accordance with Cayley's theory of algebraic invariants [10] (see also [73]). Two proper invariants, p and q (see further Section 8) were found only in 1960 by Ovsyannikov[90]. Note, that Laplace's semi-invariants and Ovsyannikov's invariants were discovered by accident. The question on existence of other invariants remained open. Thus, the problem arose on determination of all invariants for Eqs. (1.1). I called it *Laplace's problem.* The problem was solved recently in [43].

Louise Petrén, in her PhD thesis [95] defended at Lund University in 1911, extended Laplace's method and the Laplace invariants to higher-order equations. She also gave a good historical exposition which I used in the present paper, in particular, concerning Euler's priority in discovering the semi-invariants h, k. Unfortunately, her profound results remain unknown until now.

Semi-invariants for linear ordinary differential equations were intensely discussed in the 1870-1880's by J. Cockle [12], [13], E. Laguerre [72], [73], J.C. Malet [86], G.H. Halphen [24], R. Harley [25], and A.R. Forsyth [20]. The restriction to linear equations was essential in their approach. They used calculations following directly from the definition of invariants. These calculations would be extremely lengthy in the case of nonlinear equations. Indeed, when Roger Liouville [82], [83] investigated the invariants for the

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* *Author's note to this 2009 edition:* Paper 1 in this volume.

following class of nonlinear ordinary differential equations:

$$y'' + F_3(x,y)y'^{\,3} + F_2(x,y)y'^{\,2} + F_1(x,y)y' + F(x,y) = 0,$$

introduced by Lie [77], the direct method led to 70 pages of calculations.

In his review paper [79], Chapter I, §1.11, Lie noticed: "I refer to the remarkable works of Laguerre (1879) and Halphen (1882) on transformations of ordinary linear differential equations. These investigations in fact deal with the infinite group of transformations $x' = f(x), y' = g(x)y$ which is mentioned by neither of the authors. I think that Laguerre and Halphen did not know my theory." Lie himself did not have time to develop this idea. Lie's remark provided me with an incentive to begin in 1996 the systematic development of a new approach to calculation of invariants by using the infinitesimal technique, that was lacking in old methods. The method was developed in [38] ((see also [39], Chapter 10) and subsequently applied to numerous linear and nonlinear equations. These applications showed that the method is effective for determining invariants for equations with finite or infinite equivalence groups.

The present paper is a practical guide for calculation of invariants for families of linear and nonlinear differential equations with special emphasis on the use of infinite equivalence Lie algebras.

2 Two methods for calculating equivalence groups

Equivalence transformations play the central part in the theory of invariants discussed in the present paper. The set of all equivalence transformations of a given family of equations forms a group called the equivalence group and denoted by $\mathcal{E}$. The continuous group of equivalence transformations is a subgroup of $\mathcal{E}$ and is denoted by $\mathcal{E}_c$.

In this section, we discuss the notation and illustrate two main methods for calculation of equivalence transformations for families of equations. The first method consists in the direct search for the equivalence transformations and, theoretically, allows one to calculate the most general equivalence group $\mathcal{E}$. The direct method was used by Lie [78] (see also [93]) for calculation of the equivalence transformations and group classification of a family of second-order ordinary differential equations. Lie's result is discussed in Section 2.1. The direct method is further discussed in Section 3.2 for the nonlinear filtration equations.

However, the direct method leads, in general, to considerable computational difficulties. One will have the similar situation if one will calculate symmetry groups by using Lie's infinitesimal method and by the direct method.

Therefore, I employ mostly the second method suggested by Ovsyannikov [92] for determining generators of continuous equivalence groups $\mathcal{E}_c$. The central part in this method is played by what I call here a *secondary prolongation*. This concept leads to a modification of Lie's infinitesimal method. After simple introductory examples given in Section 2.2 and Section 2.4, I describe in detail the essence of the method in Section 3.1 and Section 4.1.

In what follows, the Lie algebra of the continuous equivalence group $\mathcal{E}_c$ is called the *equivalence algebra* and is denoted by $L_{\mathcal{E}}$.

2.1 Equivalence transformations for $y'' = F(x, y)$

Following Lie (see [78], §2, p. 440-446), I discus here the equivalence transformations for the following family of second-order ordinary differential equations:

$$y'' = F(x, y). \tag{2.1}$$

Definition 5.1. An equivalence transformation of the family of the equations (2.1) is a changes of variables

$$\bar{x} = \varphi(x, y), \quad \bar{y} = \psi(x, y) \tag{2.2}$$

carrying every equation of the form (2.1) into an equation of the same form:

$$\bar{y}'' = \overline{F}(\bar{x}, \bar{y}). \tag{2.3}$$

The function $\overline{F}$ may be, in general, different from the original function F. The equations (2.1) and (2.3) are said to be equivalent.

In this simple example, one can readily find the equivalence transformations by the direct method. Namely, the change of variables (2.2) implies the equations

$$\bar{y}\,' \equiv \frac{d\bar{y}}{d\bar{x}} = \frac{\psi_x + y'\psi_y}{\varphi_x + y'\varphi_y} \tag{2.4}$$

and

$$\bar{y}\,'' = \frac{\begin{vmatrix} \varphi_x + y'\varphi_y & \varphi_{xx} + 2y'\varphi_{xy} + y'^{\,2}\varphi_{yy} + y''\varphi_y \\ \psi_x + y'\psi_y & \psi_{xx} + 2y'\psi_{xy} + y'^{\,2}\psi_{yy} + y''\psi_y \end{vmatrix}}{(\varphi_x + y'\varphi_y)^3} \cdot \tag{2.5}$$

for the change of the first and second derivatives, respectively (see [78], p. 440, or [39], Section 12.3.1). Now we substitute (2.5) in Eq. (2.3) and have:

$$(\varphi_x + y'\varphi_y)^3 \overline{F}(\bar{x},\bar{y}) = \begin{vmatrix} \varphi_x + y'\varphi_y & \varphi_{xx} + 2y'\varphi_{xy} + y'^{\,2}\varphi_{yy} + F(x,y)\varphi_y \\ \psi_x + y'\psi_y & \psi_{xx} + 2y'\psi_{xy} + y'^{\,2}\psi_{yy} + F(x,y)\psi_y \end{vmatrix}.$$

Since $F(x,y)$ and $\overline{F}(\bar{x},\bar{y})$ do not depend on y', the latter equation splits into *four* equations obtained by equating to zero the coefficients for y'^3, y'^2, y' and the term without y'. Collecting the coefficients for y'^3, y'^2, y' and taking into account that $F(x,y)$, and hence $\overline{F}(\bar{x},\bar{y})$ are arbitrary functions, we get:

$$\varphi_y = 0, \quad \varphi_x\psi_{yy} = 0, \quad 2\varphi_x\psi_{xy} - \psi_y\varphi_{xx} = 0.$$

The first equation of this system yields

$$\varphi = \varphi(x),$$

where $\varphi(x)$ is and arbitrary function obeying the non-degeneracy condition $\varphi'(x) \neq 0$. The latter condition reduces the second equation $\psi_{yy} = 0$, and hence

$$\psi = \alpha(x)y + \beta(x), \quad \alpha(x) \neq 0.$$

Finally, the third equation becomes

$$\frac{a'}{a} = \frac{\varphi''}{\varphi'}\,,$$

whence

$$a(x) = A\sqrt{|\varphi'(x)|}, \quad A = \text{const}.$$

The remaining term, that does not contain y', provides the following expression for the right-hand side $\overline{F}$ of Eq. (2.3):

$$\overline{F} = \frac{A}{(\varphi')^{3/2}}\,F + A\Big[\frac{\varphi'''}{2(\varphi')^{5/2}} - \frac{3(\varphi'')^2}{4(\varphi')^{7/2}}\Big]y + \frac{\beta''}{(\varphi')^2} - \frac{\beta'\varphi''}{(\varphi')^3}\,.$$

Collecting together the above expressions for φ, ψ and $\overline{F}$, we formulate the result.

Theorem 5.1. The equivalence group $\mathcal{E}$ for the equations (2.1) is an infinite group given by the transformations

$$\bar{x} = \varphi(x), \quad \bar{y} = A\sqrt{\varphi'(x)}\,y + \beta(x), \tag{2.6}$$

$$\overline{F} = \frac{A}{(\varphi')^{3/2}}\,F + A\Big[\frac{\varphi'''}{2(\varphi')^{5/2}} - \frac{3(\varphi'')^2}{4(\varphi')^{7/2}}\Big]y + \frac{\beta''}{(\varphi')^2} - \frac{\beta'\varphi''}{(\varphi')^3}\,, \tag{2.7}$$

where $\varphi(x)$ is an arbitrary function such that $\varphi'(x) = 0$, and $A \neq 0$ is an arbitrary constant. In order to obtain the function $\overline{F}(\bar{x},\bar{y})$, it suffices to express x, y via $\bar{x}, \bar{y}$ from the equations (2.6) and substitute in (2.7).

2.2 Infinitesimal method illustrated by $y'' = F(x, y)$

Let us find the continuous group $\mathcal{E}_c$ of equivalence transformations by means of the infinitesimal method. Since the right-hand side of Eq. (2.1) may change under equivalence transformations, we treat F as a new variable and, adding to (2.2) an arbitrary transformation of F, consider the *extended transformation*:

$$\bar{x} = \varphi(x, y), \quad \bar{y} = \psi(x, y), \quad \overline{F} = \Phi(x, y, F). \tag{2.8}$$

Consequently, we look for the generator of the continuous equivalence group written in the *extended space* of variables (x, y, F) as follows:

$$Y = \xi(x, y)\frac{\partial}{\partial x} + \eta(x, y)\frac{\partial}{\partial y} + \mu(x, y, F)\frac{\partial}{\partial F}\,\cdot$$

Here y is a differential function with one independent variable x, whereas F is a differential function with two independent variables x, y. Accordingly, the prolongation of Y to y'' is given by the usual prolongation procedure, namely:

$$\widetilde{Y} = \xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y} + \mu\frac{\partial}{\partial F} + \zeta_2\frac{\partial}{\partial y''}\,,$$

where

$$\zeta_2 = \eta_{xx} + (2\eta_{xy} - \xi_{xx})y' + (\eta_{yy} - 2\xi_{xy})y'^2 - \xi_{yy}\,y'^3 + (\eta_y - 2\xi_x - 3\xi_y\,y')y''.$$

The infinitesimal invariance test of Eq. (2.1) has the form

$$\zeta_2\Big|_{y''=F} = \mu.$$

Substituting here the expression for ζ_2, we have:

$$\begin{aligned}\eta_{xx} + (2\eta_{xy} - \xi_{xx})y' + (\eta_{yy} - 2\xi_{xy})y'^2 - \xi_{yy}\,y'^3 \\ +(\eta_y - 2\xi_x - 3\xi_y\,y')F = \mu(x, y, F),\end{aligned} \tag{2.9}$$

where F is a *variable*, not a *function* $F(x, y)$. Hence, Eq. (2.9) should be satisfied identically in the independent variables x, y, y', and F. Accordingly, we split Eq. (2.9) into four equations by annulling the terms with different powers of y'. Since $\mu(x, y, F)$ does not depend on y', we obtain the following equations:

$$y'^3: \quad \xi_{yy} = 0, \tag{2.10}$$

$$y'^2: \quad \eta_{yy} - 2\xi_{xy} = 0, \tag{2.11}$$

$$y': \quad 2\eta_{xy} - \xi_{xx} - 3\xi_y F = 0, \tag{2.12}$$

$$(y')^0: \quad \mu = (\eta_y - 2\xi_x)F + \eta_{xx}. \tag{2.13}$$

Invoking that ξ and η do not depend upon F, we split Eq. (2.12) into two equations:

$$\xi_y = 0, \quad 2\eta_{xy} - \xi_{xx} = 0.$$

The first equation yields $\xi = \xi(x)$. Then the second equation is written $2\eta_{xy} = \xi'(x)$, whence upon integration:

$$\eta = \Big[\frac{1}{2}\,\xi'(x) + C\Big]y + \beta(x).$$

Now the equations (2.10)-(2.12) are manifestly satisfied, and the remaining equation (2.13) yields:

$$\mu = \Big[C - \frac{3}{2}\,\xi'(x)\Big]F + \frac{1}{2}\,\xi'''(x)y + \beta''(x), \quad C = \text{const}.$$

Thus, the general solution of equations (2.10)-(2.13) has the form

$$\xi = \xi(x), \quad \eta = \Big[\frac{1}{2}\,\xi'(x) + C\Big]y + \beta(x),$$

$$\mu = \Big[C - \frac{3}{2}\,\xi'(x)\Big]F + \frac{1}{2}\,\xi'''(x)y + \beta''(x). \tag{2.14}$$

We summarize.

Theorem 5.2. The continuous equivalence group $\mathcal{E}_c$ for Eqs. (2.1) is an infinite group. The corresponding equivalence algebra $L_{\mathcal{E}}$ is spanned by the operators

$$Y_0 = y\frac{\partial}{\partial y} + F\frac{\partial}{\partial F}\,, \quad Y_\beta = \beta(x)\frac{\partial}{\partial y} + \beta''(x)\frac{\partial}{\partial F}\,,$$

$$Y_\xi = \xi(x)\frac{\partial}{\partial x} + \frac{y}{2}\,\xi'(x)\frac{\partial}{\partial y} + \Big[\frac{y}{2}\,\xi'''(x) - \frac{3}{2}\xi'(x)F\Big]\frac{\partial}{\partial F}\,\cdot \tag{2.15}$$

Remark 5.1. Equations (2.13) can be obtained from Theorem 5.1 by letting $A > 0$, setting $\varphi(x) = x + a\xi(x)$ with a small parameter a, and writing Eqs. (2.6)-(2.7) in the first order of precision with respect to a. Hence, the transformations of the continuous equivalence group $\mathcal{E}_c$ have the form (2.6)-(2.7), where $A > 0$. Thus, the continuous equivalence group $\mathcal{E}_c$ differs from the general equivalence group $\mathcal{E}$ given by Theorem 5.1 only by the restriction $A > 0$.

2.3 Equivalence group for linear ODEs

In theory of invariants, it is advantageous to write linear homogeneous ordinary differential equations of the nth order in a *standard form* involving the binomial coefficients:

$$L_n(y) \equiv y^{(n)} + nc_1 y^{(n-1)} + \frac{n(n-1)}{2!} c_2 y^{(n-2)} + \cdots + nc_{n-1} y' + c_n y = 0, \quad (2.16)$$

where $c_i = c_i(x)$ are arbitrary variable coefficients, and $y' = dy/dx$, etc.

An equivalence transformation of the equations (2.16) is an invertible transformation (2.2) of the independent variable x and the dependent variables y preserving the order n of any equation (2.16) and its linearity and homogeneity. Recall the well-known classical result.

Theorem 5.3. The set of all equivalence transformations of the equations (2.16) is an infinite group composed of the linear transformation of the dependent variable:

$$\bar{x} = x, \quad y = \phi(x)\bar{y}, \quad (2.17)$$

where $\phi(x) \neq 0$, and an arbitrary change of the independent variable:

$$\bar{x} = f(x), \quad \bar{y} = y, \quad (2.18)$$

where $f'(x) \neq 0$.

In calculation of invariants of linear equations, we will use in Section 7 the infinitesimal form of the extension (cf. (2.8)) of each transformation (2.17) and (2.18) to the coefficients of Eq. (2.16).

Let us find the extension of the infinitesimal transformation (2.17) for the equation (2.16) of the second order,

$$L_2(y) \equiv y'' + 2c_1(x)y' + c_2(x)y = 0, \quad (2.19)$$

and for the equation of the third order,

$$L_3(y) \equiv y''' + 3c_1(x)y'' + 3c_2(x)y' + c_3(x)y = 0. \quad (2.20)$$

We implement the infinitesimal transformation (2.17) by letting

$$\phi(x) = 1 - \varepsilon\eta(x) \quad (2.21)$$

with a small parameter ε. Then

$$\begin{aligned}
y &\approx (1 - \varepsilon\eta)\,\bar{y},\\
y' &\approx (1 - \varepsilon\eta)\,\bar{y}\,' - \varepsilon\eta'\,\bar{y},\\
y'' &\approx (1 - \varepsilon\eta)\,\bar{y}\,'' - \varepsilon(2\eta'\,\bar{y}\,' + \eta''\,\bar{y}).\\
y''' &\approx (1 - \varepsilon\eta)\,\bar{y}\,''' - \varepsilon(3\eta'\,\bar{y}\,'' + 3\eta''\,\bar{y}\,' + \eta'''\,\bar{y}).
\end{aligned}$$

Substituting these expressions in Eq. (2.19), dividing by $(1-\varepsilon\eta)$ and noting that $\varepsilon/(1-\varepsilon\eta) \approx \varepsilon$, one obtains:

$$L_2(y) \approx \bar{y}\,'' + 2[c_1 - \varepsilon\eta']\,\bar{y}\,' + [c_2 - \varepsilon(\eta'' + 2c_1\eta')]\bar{y}.$$

Hence, the infinitesimal equivalence transformation (2.17) maps Eq. (2.19) into an equivalent equation:

$$\bar{y}\,'' + 2\bar{c}_1(x)\bar{y}\,' + \bar{c}_2(x)\,\bar{y} = 0, \tag{2.22}$$

where

$$\bar{c}_1 \approx c_1 - \varepsilon\eta', \quad \bar{c}_2 \approx c_2 - \varepsilon(\eta'' + 2c_1\eta'). \tag{2.23}$$

Eqs. (2.23), together with the equation $y \approx (1-\varepsilon\eta)\,\bar{y}$, provide the following generator of the equivalence transformation (2.17) extended to the coefficients of the second-order equation (2.19):

$$Y_\eta = \eta\frac{\partial}{\partial y} + \eta'\frac{\partial}{\partial c_1} + \big(\eta'' + 2c_1\eta'\big)\frac{\partial}{\partial c_2}\,\cdot \tag{2.24}$$

Likewise, the third-order equation (2.20) is transformed into an equivalent equation

$$\bar{y}\,''' + 3\bar{c}_1(x)\,\bar{y}\,'' + 3\bar{c}_2(x)\,\bar{y}\,' + \bar{c}_3(x)\,\bar{y} = 0, \tag{2.25}$$

where

$$\begin{aligned} &\bar{c}_1 \approx c_1 - \varepsilon\eta', \\ &\bar{c}_2 \approx c_2 - \varepsilon(\eta'' + 2c_1\eta'), \\ &\bar{c}_3 \approx c_3 - \varepsilon(\eta''' + 3c_1\eta'' + 3c_2\eta'). \end{aligned} \tag{2.26}$$

Eqs. (2.26), together with the equation $y \approx (1-\varepsilon\eta)\,\bar{y}$, provide the following generator of the equivalence transformation (2.17) extended to the coefficients of the third-order equation (2.20):

$$Y_\eta = \eta\frac{\partial}{\partial y} + \eta'\frac{\partial}{\partial c_1} + \big(\eta'' + 2c_1\eta'\big)\frac{\partial}{\partial c_2} + \big(\eta''' + 3c_1\eta'' + 3c_2\eta'\big)\frac{\partial}{\partial c_3}\,\cdot \tag{2.27}$$

Let us find the extension of the infinitesimal transformation (2.18) for the third-order equation (2.20). We take the infinitesimal transformation (2.20),

$$\overline{x} \approx x + \varepsilon\xi(x)$$

and have:

$$\begin{aligned} y' &\approx (1+\varepsilon\xi')\bar{y}\,', \\ y'' &\approx (1+2\varepsilon\xi')\bar{y}\,'' + \varepsilon\bar{y}\,'\xi'', \\ y''' &\approx (1+3\varepsilon\xi')\bar{y}\,''' + 3\varepsilon\bar{y}\,''\xi'' + \varepsilon\bar{y}\,'\xi'''. \end{aligned}$$

Consequently Eq. (2.20) becomes

$$\bar{y}\,''' + 3\bar{c}_1\bar{y}\,'' + 3\bar{c}_2\bar{y}\,' + \bar{c}_3\bar{y} = 0,$$

where

$$\bar{c}_1 \approx c_1 + \varepsilon(\xi'' - c_1\xi'),\ \bar{c}_2 \approx c_2 + \varepsilon\Big(\frac{1}{3}\xi''' + c_1\xi'' - 2c_2\xi'\Big),\ \bar{c}_3 \approx c_3 - 3\varepsilon c_3\xi'.$$

The corresponding group generator is

$$X_\xi = \xi\frac{\partial}{\partial x} + \big(\xi'' - c_1\xi'\big)\frac{\partial}{\partial c_1} + \Big(\frac{1}{3}\xi''' + c_1\xi'' - 2c_2\xi'\Big)\frac{\partial}{\partial c_2} - 3c_3\xi'\frac{\partial}{\partial c_3}\cdot \tag{2.28}$$

2.4 A system of linear ODEs

Let us calculate the continuous equivalence group $\mathcal{E}_c$ for the following system of linear second-order ordinary differential equations:

$$x'' + V(t)\,x = 0, \quad y'' - V(t)\,y = 0. \tag{2.29}$$

An equivalence transformation of the system (2.29) is a change of variables $t, x, y:$

$$\bar{t} = \alpha(t,x,y,a), \quad \bar{x} = \beta(t,x,y,a), \quad \bar{y} = \gamma(t,x,y,a) \tag{2.30}$$

mapping the system (2.29) into a system of the same form,

$$\frac{d^2\bar{x}}{d\bar{t}^2} + \overline{V}(\bar{t})\,\bar{x} = 0, \quad \frac{d^2\bar{y}}{d\bar{t}^2} - \overline{V}(\bar{t})\,\bar{y} = 0,$$

where the function $\overline{V}(\bar{t})$ can, in general, be different from the original function $V(t)$. Accordingly, we write the equivalence transformation (2.31) and the system (2.29) in the following *extended forms:*

$$\bar{t} = \alpha(t,x,y), \quad \bar{x} = \beta(t,x,y), \quad \bar{y} = \gamma(t,x,y), \quad \overline{V} = \Psi(t,x,y,V), \tag{2.31}$$

and

$$x'' + x\,V = 0, \quad y'' - y\,V = 0, \quad V_x = 0, \quad V_y = 0, \tag{2.32}$$

respectively. Here, x and y are, as before, the differential variables with the independent variable t, whereas V is a new differential variable with three independent variables t, x and y. Consequently, the infinitesimal generator of a one-parameter group of equivalence transformations is written in the form

$$Y = \tau(t,x,y)\frac{\partial}{\partial t} + \xi(t,x,y)\frac{\partial}{\partial x} + \eta(t,x,y)\frac{\partial}{\partial y} + \mu(t,x,y,V)\frac{\partial}{\partial V}\cdot \tag{2.33}$$

The extension of the operator (2.33) to all quantities involved in Eqs. (2.32) has the form (see [92], [1] and [67]):

$$\widetilde{Y} = \tau\frac{\partial}{\partial t} + \xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y} + \mu\frac{\partial}{\partial V} + \zeta_2^1\frac{\partial}{\partial x''} + \zeta_2^2\frac{\partial}{\partial y''} + \omega_1\frac{\partial}{\partial V_x} + \omega_2\frac{\partial}{\partial V_y}\cdot \tag{2.34}$$

The condition that Y is a generator of an equivalence group is equivalent to the statement that $\widetilde{Y}$ satisfies the infinitesimal invariance test for the extended system (2.32). This gives the following *determining equations:*

$$\zeta_2^1\Big|_{(2.32)} + \xi\,V + x\,\mu = 0, \qquad \zeta_2^2\Big|_{(2.32)} - \eta\,V - y\,\mu = 0, \tag{2.35}$$

$$\omega_1\Big|_{(2.32)} = 0, \qquad \omega_2\Big|_{(2.32)} = 0, \tag{2.36}$$

where ζ_2^1, ζ_2^2 are given by the usual prolongation procedure. Namely,

$$\zeta_2^1 = D_t(\zeta_1^1) - x''D_t(\tau), \qquad \zeta_2^2 = D_t(\zeta_1^2) - y''D_t(\tau), \tag{2.37}$$

where

$$\begin{aligned} \zeta_1^1 &= D_t(\xi) - x'D_t(\tau) = \xi_t + x'\xi_x + y'\xi_y - x'(\tau_t + x'\tau_x + y'\tau_y), \\ \zeta_1^2 &= D_t(\eta) - y'D_t(\tau) = \eta_t + x'\eta_x + y'\eta_y - y'(\tau_t + x'\tau_x + y'\tau_y). \end{aligned} \tag{2.38}$$

The coefficients ω_1 and ω_2 are determined by

$$\begin{aligned} \omega_1 &= \widetilde{D}_x(\mu) - V_x\widetilde{D}_x(\xi) - V_y\widetilde{D}_x(\eta) - V_t\widetilde{D}_x(\tau), \\ \omega_2 &= \widetilde{D}_y(\mu) - V_x\widetilde{D}_y(\xi) - V_y\widetilde{D}_y(\eta) - V_t\widetilde{D}_y(\tau). \end{aligned} \tag{2.39}$$

We used here the notation

$$D_t = \frac{\partial}{\partial t} + x'\frac{\partial}{\partial x} + y'\frac{\partial}{\partial y} + x''\frac{\partial}{\partial x'} + y''\frac{\partial}{\partial y'}$$

for the usual total differentiation with respect to t, while

$$\widetilde{D}_x = \frac{\partial}{\partial x} + V_x\frac{\partial}{\partial V}\,, \quad \widetilde{D}_y = \frac{\partial}{\partial y} + V_y\frac{\partial}{\partial V} \tag{2.40}$$

denote the "new" total differentiations for the extended system (2.32).

The restriction to Eqs. (2.32) means, in particular, that we set $V_x = V_y = 0$. Then the expressions (2.40) and (2.39) take the form:

$$\widetilde{D}_x = \frac{\partial}{\partial x}\,, \quad \omega_1 = \mu_x - V_t \tau_x,$$
$$\widetilde{D}_y = \frac{\partial}{\partial y}\,, \quad \omega_2 = \mu_y - V_t \tau_y.$$

Let us solve the determining equations. We begin with the equations (2.36):

$$\omega_1 \equiv \mu_x - V_t \tau_x = 0, \quad \omega_2 \equiv \mu_y - V_t \tau_y = 0.$$

Since V and hence V_t are arbitrary functions, the above equations yield:

$$\tau_x = 0, \quad \mu_x = 0, \quad \tau_y = 0, \quad \mu_y = 0.$$

Thus, the operator (2.33) reduces to the form

$$Y = \tau(t)\frac{\partial}{\partial t} + \xi(t,x,y)\frac{\partial}{\partial x} + \eta(t,x,y)\frac{\partial}{\partial y} + \mu(t,V)\frac{\partial}{\partial V}\,\cdot \tag{2.41}$$

Let us turn now to the remaining determining equations (2.35). We apply to the operator (2.41) the prolongation formulae (2.37) and substitute the resulting expression for ζ_2^1 in the first equation (2.35) to obtain:

$$\begin{aligned} &\xi_{tt} + (2\xi_{tx} - \tau'')x' + 2\xi_{ty}y' + \xi_{xx}\,x'^2 + 2\xi_{xy}\,x'y' \\ &+\xi_{yy}y'^2 + (y\xi_y - x\xi_x + 2x\tau' + \xi)V + x\mu = 0. \end{aligned} \tag{2.42}$$

We collect here the like terms and annul the coefficients of different powers of x' and y'. The coefficients for $x'^2, x'y', y'^2$, and y' yield:

$$\xi_{xx} = 0, \quad \xi_{xy} = 0, \quad \xi_{yy} = 0, \quad \xi_{ty} = 0,$$

whence

$$\xi = a(t)x + Ay + k(t), \quad A = \text{const}.$$

Furthermore, annulling the coefficient $2\xi_{tx} - \tau'' = 0$ for x' we have $2a'(t) = \tau''(t)$, and hence

$$a(t) = \frac{1}{2}\tau'(t) + C_1.$$

Thus,

$$\xi = \Big(\frac{1}{2}\tau'(t) + K_1\Big)x + Ay + k(t). \tag{2.43}$$

Now Eq. (2.42) becomes

$$\frac{x}{2}\tau'''(t) + k''(t) + [2Ay + k(t) + 2x\tau'(t)]V + x\mu = 0. \tag{2.44}$$

Likewise, the second equation (2.35) yields

$$\eta = \Big(\frac{1}{2}\tau'(t) + K_2\Big)y + Bx + l(t) \tag{2.45}$$

and

$$\frac{y}{2}\tau'''(t) + l''(t) - [2Bx + l(t) + 2y\tau'(t)]V - y\mu = 0. \tag{2.46}$$

Since V is regarded as an arbitrary variable, and μ does not depend upon x and y, Eq. (2.44) yields $A = 0,\ k(t) = 0$ and

$$\mu = -\frac{1}{2}\tau'''(t) - 2\tau'(t)V. \tag{2.47}$$

Likewise, Eq. (2.46) yields $B = 0,\ l(t) = 0$ and

$$\mu = \frac{1}{2}\tau'''(t) - 2\tau'(t)V. \tag{2.48}$$

Eqs. (2.47) and (2.48) yield that $\tau'''(t) = 0,\ \mu = -2\tau'(t)V$. Summing up, we obtain :

$$\begin{aligned} &\tau(t) = C_3 + C_4 t + C_5 t^2,\\ &\xi = (C_1 + C_5 t)x,\\ &\eta = (C_2 + C_5 t)y,\\ &\mu = -2(C_4 + 2C_5 t)V. \end{aligned} \tag{2.49}$$

We summarize.

Theorem 5.4. The equivalence algebra $L_{\mathcal{E}}$ for the system (2.29) is a five-dimensional Lie algebra spanned by

$$\begin{aligned} &Y_1 = x\frac{\partial}{\partial x}\,, \quad Y_2 = y\frac{\partial}{\partial y}\,, \quad Y_3 = t\frac{\partial}{\partial t} - 2V\frac{\partial}{\partial V}\,,\\ &Y_4 = \frac{\partial}{\partial t}\,, \quad Y_5 = t^2\frac{\partial}{\partial t} + tx\frac{\partial}{\partial x} + ty\frac{\partial}{\partial y} - 4tV\frac{\partial}{\partial V}\,. \end{aligned} \tag{2.50}$$

Note that the operator Y_5 generates the one-parameter group of transformations

$$\overline{t} = \frac{t}{1-at}\,, \quad \overline{x} = \frac{x}{1-at}\,, \quad \overline{y} = \frac{y}{1-at}\,, \quad \overline{V} = (1-at)^4\, V. \tag{2.51}$$

3 Equivalence group for filtration equation

In this section, we discuss both methods for calculation of equivalence transformations for partial differential equations by considering the nonlinear filtration equation

$$v_t = h(v_x)v_{xx}. \tag{3.1}$$

Equation (3.1) is used in mechanics as a mathematical model in studying shear currents of nonlinear viscoplastic media, processes of filtration of non-Newtonian fluids, as well as for describing the propagation of oscillations of temperature and salinity to depths in oceans (see, e.g. [1], Chapter 2, and the references therein). The function $h(v_x)$ is known as a filtration coefficient. In general, the filtration coefficient is not fixed, and we consider the family of equations of the form (3.1) with an arbitrary function $h(v_x)$.

Definition 5.2. An equivalence transformation of the family of equations of the form (3.1) is a changes of variables

$$\bar{x} = \varphi(t, x, v), \quad \bar{t} = \psi(t, x, v), \quad \bar{v} = \Phi(t, x, v) \tag{3.2}$$

carrying every equation of the form (3.1) with any filtration coefficient $h(v_x)$ into an equation of the same form:

$$\bar{v}_{\bar{t}} = \bar{h}(\bar{v}_{\bar{x}})\bar{v}_{\bar{x}\bar{x}}. \tag{3.3}$$

The function $\bar{h}$ representing a new filtration coefficient $\bar{h}(\bar{v}_{\bar{x}})$ may be, in general, different from the original function h.

3.1 Secondary prolongation and infinitesimal method

Equation (3.1) provides a good example for introducing the concept of a *secondary prolongation* and illustrating the infinitesimal method for calculating the continuous equivalence group $\mathcal{E}_c$.

In order to find the continuous group $\mathcal{E}_c$ of equivalence transformations (3.2), we search for the generators of the group $\mathcal{E}_c$:

$$Y = \xi^1 \frac{\partial}{\partial t} + \xi^2 \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial v} + \mu \frac{\partial}{\partial h} \cdot \tag{3.4}$$

The generator Y defines the group $\mathcal{E}_c$ of equivalence transformations

$$\bar{x} = \varphi(t, x, v), \quad \bar{t} = \psi(t, x, v), \quad \bar{v} = \Phi(t, x, v), \quad \bar{h} = F(t, x, v, v_t, v_x, h) \tag{3.5}$$

for the family of equations (3.1) if and only if Y obeys the condition of invariance of the following system:

$$v_t = hv_{xx}, \quad h_t = 0, \quad h_x = 0, \quad h_v = 0, \quad h_{v_t} = 0. \tag{3.6}$$

In order to write the infinitesimal invariance test for the system (3.6), we should extend the action of the operator (3.4) to all variables involved in (3.6), i.e. take

$$\widetilde{Y} = Y + \zeta_1 \frac{\partial}{\partial v_t} + \zeta_2 \frac{\partial}{\partial v_x} + \zeta_{22} \frac{\partial}{\partial v_{xx}} + \omega_1 \frac{\partial}{\partial h_t} + \omega_2 \frac{\partial}{\partial h_x} + \omega_3 \frac{\partial}{\partial h_v} + \omega_4 \frac{\partial}{\partial h_{v_t}} \cdot$$

First, we extend Y to the derivatives of v_t, v_x and v_{xx} by treating v as a differential variable depending on (t, x), as we do in Eq. (3.1). The unknown coordinates ξ^1, ξ^2 and η of the operator (3.4) are sought as functions of the variables t, x, v. Accordingly, we use the usual total differentiations in the space (t, x, v) :

$$D_1 \equiv D_t = \frac{\partial}{\partial t} + v_t \frac{\partial}{\partial v} + v_{tt} \frac{\partial}{\partial v_t} + v_{tx} \frac{\partial}{\partial v_x} \,,$$

$$D_2 \equiv D_x = \frac{\partial}{\partial x} + v_x \frac{\partial}{\partial v} + v_{tx} \frac{\partial}{\partial v_t} + v_{xx} \frac{\partial}{\partial v_x}$$

and calculate the coordinates ζ_1, ζ_2 and ζ_{22} by the usual prolongation formulae:

$$\zeta_i = D_i(\eta) - v_t D_i(\xi^1) - v_x D_i(\xi^2), \tag{3.7}$$

$$\zeta_{22} = D_2(\zeta_2) - v_{tx} D_2(\xi^1) - v_{xx} D_2(\xi^2). \tag{3.8}$$

Then we pass to the extended space (t, x, v, v_t, v_x) and consider h as a differential variable depending on the independent variables (t, x, v, v_t, v_x). The crucial step of the secondary prolongation is that we consider the coordinate μ of the equivalence operator (3.4) as a function of t, x, v, v_t, v_x, h and introduce the new total differentiations in the extended space t, x, v, v_t, v_x, h :

$$\begin{aligned}
\widetilde{D}_1 \equiv \widetilde{D}_t &= \frac{\partial}{\partial t} + h_t \frac{\partial}{\partial h} + h_{tt} \frac{\partial}{\partial h_t} + h_{tx} \frac{\partial}{\partial h_x} + h_{tv} \frac{\partial}{\partial h_v} + h_{tv_t} \frac{\partial}{\partial h_{v_t}} \,, \\
\widetilde{D}_2 \equiv \widetilde{D}_x &= \frac{\partial}{\partial x} + h_x \frac{\partial}{\partial h} + h_{xt} \frac{\partial}{\partial h_t} + h_{xx} \frac{\partial}{\partial h_x} + h_{xv} \frac{\partial}{\partial h_v} + h_{xv_t} \frac{\partial}{\partial h_{v_t}} \,, \\
\widetilde{D}_3 \equiv \widetilde{D}_v &= \frac{\partial}{\partial v} + h_v \frac{\partial}{\partial h} + h_{vt} \frac{\partial}{\partial h_t} + h_{vx} \frac{\partial}{\partial h_x} + h_{vv} \frac{\partial}{\partial h_v} + h_{vv_t} \frac{\partial}{\partial h_{v_t}} \,, \qquad (3.9) \\
\widetilde{D}_4 \equiv \widetilde{D}_{v_t} &= \frac{\partial}{\partial v_t} + h_{v_t} \frac{\partial}{\partial h} + h_{tv_t} \frac{\partial}{\partial h_t} + h_{xv_t} \frac{\partial}{\partial h_x} + h_{vv_t} \frac{\partial}{\partial h_v} + h_{v_t v_t} \frac{\partial}{\partial h_{v_t}} \cdot
\end{aligned}$$

Then we use the result of the *usual prolongation* (3.8), specifically the expression for ζ_2, and calculate the coordinates ω_i by the following *new prolongation formulae*:

$$\omega_i = \widetilde{D}_i(\mu) - h_t\widetilde{D}_i(\xi^1) - h_x\widetilde{D}_i(\xi^2) - h_v\tilde{D}_i(\eta) - h_{v_t}\widetilde{D}_i(\zeta_1) - h_{v_x}\widetilde{D}_i(\zeta_2). \quad (3.10)$$

The infinitesimal invariance test for the system (3.6) has the form

$$\Big(\zeta_1 - h\zeta_{22} - \mu v_{xx}\Big)\Big|_{(3.6)} = 0, \quad (3.11)$$

$$\omega_i\Big|_{(3.6)} = 0, \quad i = 1, \ldots, 4. \quad (3.12)$$

Taking into account the equations (3.6) and invoking that μ does not depend on the derivatives $h_x, \ldots, h_{v_t}$, we reduce the differentiations (3.9) to the following partial derivatives:

$$\widetilde{D}_1 = \frac{\partial}{\partial t}\,, \quad \widetilde{D}_2 = \frac{\partial}{\partial x}\,, \quad \widetilde{D}_3 = \frac{\partial}{\partial v}\,, \quad \widetilde{D}_3 = \frac{\partial}{\partial v_t}\,\cdot$$

This simplifies the prolongation formulae (3.10). Since the functions ξ, η, ζ do not depend on h, the equations (3.12) become

$$\frac{\partial \mu}{\partial t} - h_{v_x}\frac{\partial \zeta_2}{\partial t} = 0, \quad \frac{\partial \mu}{\partial x} - h_{v_x}\frac{\partial \zeta_2}{\partial x} = 0,$$
$$\frac{\partial \mu}{\partial v} - h_{v_x}\frac{\partial \zeta_2}{\partial v} = 0, \quad \frac{\partial \mu}{\partial v_t} - h_{v_x}\frac{\partial \zeta_2}{\partial v_t} = 0.$$

Since h and h_{v_x} are algebraically independent, the above equations split into the following two systems:

$$\frac{\partial \mu}{\partial t} = 0, \quad \frac{\partial \mu}{\partial x} = 0, \quad \frac{\partial \mu}{\partial v} = 0, \quad \frac{\partial \mu}{\partial v_t} = 0 \quad (3.13)$$

and

$$\mu = \mu(v_x, h),$$
$$\frac{\partial \zeta_2}{\partial t} = 0, \frac{\partial \zeta_2}{\partial x} = 0, \frac{\partial \zeta_2}{\partial v} = 0, \frac{\partial \zeta_2}{\partial v_t} = 0. \quad (3.14)$$

Equations (3.13) yield

$$\mu = \mu(v_x, h). \quad (3.15)$$

Substituting the the expression $\zeta_2 = \eta_x + v_x\eta_v - v_t\xi^1_x - v_t\xi^1_x - v_tv_x\xi^1_v - v_x\xi^2_x - v_x^2\xi^2_v$ (see (3.8)) in (3.14), equating to zero separately the coefficients for different powers in v_x and v_t, and integrating the resulting equations, we obtain:

$$\xi^1 = \xi^1(t), \quad \xi^2 = A_1(t)x + C_1v + A_2(t), \quad \eta = A_3(t)v + C_2x + A_4(t), \quad (3.16)$$

where $A_i(t)$ are arbitrary functions and C_i = const. Substitution of the expressions (3.15) and (3.16) into (3.11) yields the following general solution to the determining equations (3.11)-(3.12):

$$\xi^1 = C_1 t + C_2, \quad \xi^2 = C_3 x + C_4 v + C_5, \tag{3.17}$$
$$\eta = C_6 x + C_7 v + C_8. \quad \mu = (2C_4 v_x + 2C_3 - C - 1)h.$$

Substituting (3.17) in (3.4), one arrives at the following theorem.

Theorem 5.5. The equivalence algebra $L_{\mathcal{E}}$ for the filtration equations (3.1) is an 8-dimensional Lie algebra spanned by

$$Y_1 = \frac{\partial}{\partial t}, \quad Y_2 = \frac{\partial}{\partial x}, \quad Y_3 = \frac{\partial}{\partial v}, \quad Y_4 = t\frac{\partial}{\partial t} - h\frac{\partial}{\partial h}, \tag{3.18}$$
$$Y_5 = x\frac{\partial}{\partial x} + 2h\frac{\partial}{\partial h}, \quad Y_6 = v\frac{\partial}{\partial x} + 2v_x h\frac{\partial}{\partial h}, \quad Y_7 = x\frac{\partial}{\partial v}, \quad Y_8 = v\frac{\partial}{\partial v} \cdot$$

The operators (3.18) generate an eight-parameter group. The transformations of this group are obtained by solving the Lie equations for each of the basic generators (3.18) and taking the composition of the resulting one-parameter groups. Note that since the operator Y_6 involves the derivative v_x, we should use its first prolongation, namely, extend its action to v_x and write as follows:

$$Y_6 = v\frac{\partial}{\partial x} - v_x^2\frac{\partial}{\partial v_x} + 2v_x h\frac{\partial}{\partial h}$$

Then, denoting by a_6 the parameter of the one-parameter group with the generator Y_6, we have the following Lie equations:

$$\frac{d\bar{t}}{da_6} = 0, \quad \frac{d\bar{x}}{da_6} = \bar{v}, \quad \frac{d\bar{v}}{da_6} = 0, \quad \frac{d\bar{v}_{\bar{x}}}{da_6} = -\bar{v}_{\bar{x}}^2, \quad \frac{d\bar{h}}{da_6} = 2\bar{v}_{\bar{x}}\,\bar{h}.$$

The integration, using the initial conditions

$$\bar{t}|_{a_6=0} = t, \quad \bar{x}|_{a_6=0} = x, \quad \bar{v}|_{a_6=0} = v, \quad \bar{v}_{\bar{x}}|_{a_6=0} = v_x, \quad \bar{h}|_{a_6=0} = h$$

yields:

$$\bar{t} = t, \quad \bar{x} = x + va_6, \quad \bar{v} = v, \quad \bar{v}_{\bar{x}} = \frac{v_x}{1 + a_6 v_x}, \quad \bar{h} = (1 + a_6 v_x)^2\,h.$$

Whence, ignoring the transformation formula for v_x, one obtains the equivalence transformation of the form (3.5). For all other operators (3.18), the integration of the Lie equations is straightforward. We have:

$$
\begin{aligned}
&Y_1: \quad \bar{t}=t+a_1, \quad \bar{x}=x, \quad \bar{v}=v, \quad \bar{h}=h;\\
&Y_2: \quad \bar{t}=t, \quad \bar{x}=x+a_2, \quad \bar{v}=v, \quad \bar{h}=h;\\
&Y_3: \quad \bar{t}=t, \quad \bar{x}=x, \quad \bar{v}=v+a_3, \quad \bar{h}=h;\\
&Y_4: \quad \bar{t}=a_4\,t, \quad \bar{x}=x, \quad \bar{v}=v, \quad \bar{h}=\frac{1}{a_4}\,h\,, \quad a_4>0;\\
&Y_5: \quad \bar{t}=t, \quad \bar{x}=a_5\,x, \quad \bar{v}=v, \quad \bar{h}=a_5^2\,h, \quad a_5>0;\\
&Y_6: \quad \bar{t}=t, \quad \bar{x}=x+a_6\,v, \quad \bar{v}=v, \quad \bar{h}=(1+a_6v_x)^2\,h;\\
&Y_7: \quad \bar{t}=t, \quad \bar{x}=x, \quad \bar{v}=v+a_7\,x, \quad \bar{h}=h;\\
&Y_8: \quad \bar{t}=t, \quad \bar{x}=x, \quad \bar{v}=a_8\,v, \quad \bar{h}=h, \quad a_8>0.
\end{aligned}
$$

Taking the composition of these transformations and setting

$$
\begin{aligned}
&\alpha=a_4, \quad \beta_1=a_5, \quad \beta_2=a_6, \quad \beta_3=a_5a_7a_8, \quad \beta_4=(1+a_6a_7)a_8,\\
&\gamma_1=a_1a_4, \quad \gamma_2=a_2a_5+a_3a_6, \quad \gamma_3=(a_3+a_2a_5a_7+a_3a_6a_7)a_8,
\end{aligned}
$$

we arrive at the following result.

Theorem 5.6. The continuous group $\mathcal{E}_c$ of equivalence transformations (3.5) for the family of filtration equations (3.1) has the following form:

$$
\begin{aligned}
&\bar{t}=\alpha t+\gamma_1, \quad \bar{x}=\beta_1x+\beta_2v+\gamma_2,\\
&\bar{v}=\beta_3x+\beta_4v+\gamma_3, \quad \bar{h}=(\beta_1+\beta_2v_x)^2\,\frac{h}{\alpha}\,,
\end{aligned}
\tag{3.19}
$$

where α,β and γ are constant coefficients obeying the conditions

$$
\alpha>0, \quad \beta_1>0, \quad \beta_4>0, \quad \beta_1\beta_4-\beta_2\beta_3>0. \tag{3.20}
$$

Note that the last inequality in (3.20) follows from the equation $\beta_1\beta_4-\beta_2\beta_3=a_5a_8$.

3.2 Direct search for equivalence group $\mathcal{E}$

Let us outline the direct method. We look for the general equivalence transformations in the form (3.2):

$$
\bar{x}=\varphi(t,x,v), \quad \bar{t}=\psi(t,x,v), \quad \bar{v}=\Phi(t,x,v).
$$

Under this change of variables, the differentiation operators D_t, D_x are transformed according to the formulas

$$D_t = D_t(\varphi)D_{\bar{x}} + D_t(\psi)D_{\bar{t}}, \quad D_x = D_x(\varphi)D_{\bar{x}} + D_x(\psi)D_{\bar{t}},$$

the use of which leads to the following expression for $\bar{v}_{\bar{x}}$:

$$\bar{v}_{\bar{x}} = \frac{D_x(\Phi)D_t(\Psi) - D_t(\Phi)D_x(\psi)}{D_x(\varphi)D_t(\psi) - D_t(\varphi)D_x(\psi)}\,.$$

It follows from the definition of equivalence transformations that the right-hand side of the latter equation should depend only on v_x so its derivatives with respect to t, x, v, v_t are equal to zero. This condition leads to a system of equations on the functions φ, ψ, Φ whose solution has the form

$$\begin{aligned} &\varphi = A_1(t)(\beta_1 x + \beta_2 v) + A_2(t),\\ &\psi = \psi(t),\\ &\Phi = A_1(t)(\beta_3 x + \beta_4 v) + A_3(t), \end{aligned} \tag{3.21}$$

where $\beta_1\beta_4 - \beta_2\beta_3 \neq 0, \psi'(t) \neq 0, A_1(t) \neq 0.$

One can further specify the functions $A_i(t)$ and $\psi(t)$ by substituting (3.21) in Eqs. (3.2)-(3.3) and prove the following statement.

Theorem 5.7. The general equivalence group $\mathcal{E}$ of filtration equations (3.1) has the form (3.19):

$$\bar{t} = \alpha t + \gamma_1, \quad \bar{x} = \beta_1 x + \beta_2 v + \gamma_2,$$

$$\bar{v} = \beta_3 x + \beta_4 v + \gamma_3, \quad \bar{h} = (\beta_1 + \beta_2 v_x)^2\,\frac{h}{\alpha}$$

with arbitrary coefficients $\alpha, \beta_i, \gamma_i$, obeying only the the non-degeneracy condition (cf. the conditions (3.20)):

$$\beta_1\beta_4 - \beta_2\beta_3 \neq 0. \tag{3.22}$$

Remark 5.2. Theorems 5.6 and 5.7 show that the general equivalence group $\mathcal{E}$ can be obtained from the continuous equivalence group $\mathcal{E}_c$ merely by completing the latter by the reflections $t \to -t$ and $x \to -x$.

3.3 Two equations related to filtration equation

If we differentiate both sides of the filtration equation (3.1) with respect to x and set

$$u = v_x \tag{3.23}$$

we get the nonlinear heat equation

$$u_t = [h(u)u_x]_x. \tag{3.24}$$

The equivalence algebra $L_\mathcal{E}$ for Eq. (3.24) is a 6-dimensional Lie algebra spanned by following generators (cf. (3.18)):

$$Y_1 = \frac{\partial}{\partial t}, \quad Y_2 = \frac{\partial}{\partial x}, \quad Y_3 = t\frac{\partial}{\partial t} - h\frac{\partial}{\partial h}, \tag{3.25}$$

$$Y_4 = x\frac{\partial}{\partial x} + 2h\frac{\partial}{\partial h}, \quad Y_5 = \frac{\partial}{\partial u}, \quad Y_6 = u\frac{\partial}{\partial u} \cdot$$

Calculating the transformations of the continuous equivalence group $\mathcal{E}_c$ generated by (3.25) and adding the reflections

$$t \to -t, \quad x \to -x, \quad w \to -w,$$

we arrive at the well-known equivalence group $\mathcal{E}$ for Equation (3.24):

$$\begin{gathered} \tilde{t} = \alpha t + \gamma_1, \quad \tilde{x} = \beta_1 x + \gamma_2, \\ \tilde{u} = \delta_1 u + \delta_2, \quad \tilde{h} = \frac{\beta_1^2}{\alpha} h, \end{gathered} \tag{3.26}$$

where $\alpha\beta_1\delta_1 \neq 0$.

Furthermore, by setting

$$v = w_x \tag{3.27}$$

and integrating the filtration equation (3.1) with respect to x we get the equation

$$w_t = H(w_2) \tag{3.28}$$

The equivalence algebra $L_\mathcal{E}$ for Eq. (3.28) is a 9-dimensional Lie algebra spanned by

$$Y_1 = \frac{\partial}{\partial t}, \quad Y_2 = \frac{\partial}{\partial x}, \quad Y_3 = \frac{\partial}{\partial w}, \quad Y_4 = t\frac{\partial}{\partial t} - h\frac{\partial}{\partial h}, \quad Y_5 = x\frac{\partial}{\partial x}, \tag{3.29}$$

$$Y_6 = t\frac{\partial}{\partial w} + \frac{\partial}{\partial H}, \quad Y_7 = w\frac{\partial}{\partial w} + H\frac{\partial}{\partial H}, \quad Y_8 = x\frac{\partial}{\partial w}, \quad Y_9 = x^2\frac{\partial}{\partial w} \cdot$$

Calculating the transformations of the continuous equivalence group $\mathcal{E}_c$ generated by (3.29) and adding the reflections $t \to -t, x \to -x, w \to -w$, we arrive at the following complete equivalence group $\mathcal{E}$ for Equation (3.28):

$$\begin{gathered} \tilde{t} = \alpha t + \gamma_1, \quad \tilde{x} = \beta_1 x + \gamma_2, \\ \tilde{w} = \delta_1 w + \delta_2 x^2 + \delta_3 x + \delta_4 t + \delta_5, \quad H = \frac{\delta_1}{\alpha} H + \delta_4, \end{gathered} \tag{3.30}$$

where $\alpha\beta_1\delta_1 \neq 0$.

4 Equivalence groups for nonlinear wave equations

4.1 Equations $v_{tt} = f(x, v_x)v_{xx} + g(x, v_x)$

Consider the family of nonlinear wave equations (see [67])

$$v_{tt} = f(x, v_x)v_{xx} + g(x, v_x) \tag{4.1}$$

with two arbitrary functions $f(x, v_x)$ and $g(x, v_x)$.

Let us denote $f = f^1, g = f^2$ and seek the operator of the equivalence group $\mathcal{E}_c$ in the form

$$Y = \xi^1 \frac{\partial}{\partial t} + \xi^2 \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial v} + \mu^k \frac{\partial}{\partial f^k} \cdot \tag{4.2}$$

We write Eq. (4.1) as the system

$$v_{tt} - f^1 v_{xx} - f^2 = 0, \quad f^k_t = f^k_v = f^k_{v_t} = 0 \tag{4.3}$$

The additional equations $f^k_t = f^k_v = f^k_{v_t} = 0$ indicate that the arbitrary function f, g do not depend on t, v, v_t.

In these calculations v and f^k are considered as differential variables: v on the space (t, x) and f^k on the extended space (t, x, v, v_t, v_x). The coordinates ξ^1, ξ^2, η of the operator (4.2) are sought as functions of t, x, v while the coordinates, μ^k are sought as functions of $t, x, v, v_t, v_x, f^1, f^2$. The invariance conditions of the system (4.3) are written

$$\widetilde{Y}(v_{tt} - f^1 v_{xx} - f^2) = 0 \tag{4.4}$$

$$\tilde{Y}(f^k_t) = \tilde{Y}(f^k_v) = \tilde{Y}(f^k_{v_t}) = 0 \quad (k = 1, 2), \tag{4.5}$$

where $\tilde{Y}$ is the prolongation of the operator (4.2) to the quantities involved in Eqs. (4.3). Namely,

$$\widetilde{Y} = Y + \zeta_1 \frac{\partial}{\partial v_t} + \zeta_2 \frac{\partial}{\partial v_x} + \zeta_{11} \frac{\partial}{\partial v_{tt}} + \zeta_{22} \frac{\partial}{\partial v_{xx}} + \omega^k_1 \frac{\partial}{\partial f^k_t} + \omega^k_0 \frac{\partial}{\partial f^k_v} + \omega^k_{01} \frac{\partial}{\partial f^k_{v_t}} \cdot$$

The coefficients ζ are given by the usual prolongation formulae:

$$\begin{aligned}
\zeta_1 &= D_t(\eta) - v_t D_t(\xi^1) - v_x D_t(\xi^2), \\
\zeta_2 &= D_x(\eta) - v_t D_x(\xi^1) - v_x D_x(\xi^2), \\
\zeta_{11} &= D_t(\zeta_1) - v_{tt} D_t(\xi^1) - v_{tx} D_t(\xi^2), \\
\zeta_{22} &= D_x(\zeta_2) - v_{tx} D_x(\xi^1) - v_{xx} D_x(\xi^2),
\end{aligned}$$

whereas the coefficients ω are obtained by applying the secondary prolongation procedure (see Section 3.1) to the differential variables f^k with the independent variables (t, x, v, v_t, v_x). Namely,

$$\omega_1^k = \tilde{D}_t(\mu^k) - f_t^k \tilde{D}_t(\xi^1) - f_x^k \tilde{D}_t(\xi^2) - f_v^k \tilde{D}_t(\eta) - f_{v_t}^k \tilde{D}_t(\zeta_1) - f_{v_x}^k \tilde{D}_t(\zeta_2), \quad (4.6)$$

where $\tilde{D}_t$ has the form

$$\tilde{D}_t = \frac{\partial}{\partial t} + f_t^k \frac{\partial}{\partial f^k}$$

and inn view of Eqs. (4.3) reduces to

$$\tilde{D}_t = \frac{\partial}{\partial t} \cdot$$

Furthermore, we obtain ω_0^k and ω_{01}^k by replacing in Eq. (4.6) the operator $\tilde{D}_t$ successively by the operators

$$\tilde{D}_v = \frac{\partial}{\partial v} + f_v^k \frac{\partial}{\partial f^k}$$

and

$$\tilde{D}_{v_t} = \frac{\partial}{\partial v_t} + f_{v_t}^k \frac{\partial}{\partial f^k}$$

and noting that in view of Eqs. (4.3) we have

$$\tilde{D}_v = \frac{\partial}{\partial v}\,, \quad \tilde{D}_{v_t} = \frac{\partial}{\partial v_t} \cdot$$

Finally, we have:

$$\begin{gathered} \omega_1^k = \mu_t^k - f_x^k \xi_t^2 - f_{v_x}^k (\zeta_2)_t, \\ \omega_0^k = \mu_v^k - f_x^k \xi_v^2 - f_{v_x}^k (\zeta_2)_v, \\ \omega_{01}^k = \mu_{v_t}^k - f_{v_x}^k (\zeta_2)_{v_t}. \end{gathered} \quad (4.7)$$

The invariance conditions (4.5) have the form

$$\omega_1^k = \omega_0^k = \omega_{01}^k = 0, \quad k = 1, 2. \quad (4.8)$$

Substituting the expressions (4.7) and noting that Eqs. (4.8) hold for arbitrary values of f^1 and f^2, we obtain

$$\mu_t^k = \mu_v^k = \mu_{v_t}^k = 0, \quad \xi_t^2 = \xi_v^2 = 0,$$

$$(\zeta_2)_t = (\zeta_2)_v = (\zeta_2)v_t = 0.$$

Integration yields:

$$\begin{aligned} &\xi^1 = \xi^1(t), \quad \xi^2 = \xi^2(x) \\ &\eta = c_1 v + F(x) + H(t), \\ &\mu^k = \mu^k(x, v_x, f^1, f^2). \end{aligned} \tag{4.9}$$

Now we write the invariance condition (4.4):

$$\zeta_{11} - \mu^1 v_{xx} - f^1 \zeta_{22} - \mu^2 = 0,$$

take into account Eqs. (4.9) and obtain:

$$\begin{aligned} &(\xi^1)'' v_t + \{[C_1 - 2(\xi^1)']f^1 - \mu^1 - [C_1 - 2(\xi^2)']f^1\}v_{xx} \\ &+[C_1 - 2(\xi^1)']f^2 + H'' - f^1 F'' + f^1 v_x (\xi^2)'' - \mu^2 = 0. \end{aligned}$$

Since v, v_t, v_x and v_{xx} are independent variables, it follows:

$$\begin{aligned} &\xi^1 = C_2 t + C_3, \quad \xi^2 = \varphi(x), \\ &\eta = C_1 v + F(x) + C_4 t^2 + C_5 t, \\ &\mu^1 = 2(\varphi' - C_2) f, \\ &\mu^2 = (C_1 - 2C_2) g + 2C_4 + (\varphi'' v_x - F'') f, \end{aligned} \tag{4.10}$$

where C_1, C_2, C_3, C_4, C_5 are arbitrary constants, and $\varphi(x)$ and $F(x)$ arbitrary functions. Thus, we have the following result.

Theorem 5.8. The family of nonlinear wave equations (4.1) has an infinite equivalence group $\mathcal{E}_c$. The corresponding Lie algebra $L_{\mathcal{E}}$ is spanned by the generators

$$\begin{aligned} &Y_1 = \frac{\partial}{\partial t}, \quad Y_2 = \frac{\partial}{\partial v}, \quad Y_3 = t\frac{\partial}{\partial v}, \\ &Y_4 = x\frac{\partial}{\partial v}, \quad Y_5 = t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + 2v\frac{\partial}{\partial v}, \\ &Y_6 = t\frac{\partial}{\partial t} - 2f\frac{\partial}{\partial f} - 2g\frac{\partial}{\partial g}, \quad Y_7 = t^2\frac{\partial}{\partial v} + 2\frac{\partial}{\partial g}, \\ &Y_\varphi = \varphi(x)\frac{\partial}{\partial x} + 2\varphi'(x) f\frac{\partial}{\partial f} + \varphi''(x) v_x f\frac{\partial}{\partial g}, \\ &Y_F = F(x)\frac{\partial}{\partial v} - F''(x) f\frac{\partial}{\partial g}\cdot \end{aligned} \tag{4.11}$$

Remark 5.3. The general equivalence group $\mathcal{E}$ contains, along with the continuous subgroup $\mathcal{E}_c$, also three independent reflections:

$$t \mapsto -t, \tag{4.12}$$

$$x \mapsto -x, \tag{4.13}$$

$$v \mapsto -v, \quad g \mapsto -g. \tag{4.14}$$

4.2 Equations $u_{tt} - u_{xx} = f(u, u_t, u_x)$

Similar calculations for the nonlinear wave equations of the form

$$u_{tt} - u_{xx} = f(u, u_t, u_x) \qquad (f \neq 0). \tag{4.15}$$

provide the infinite-dimensional equivalence algebra $L_{\mathcal{E}}$ spanned by ([104])

$$Y_1 = \frac{\partial}{\partial t}, \quad Y_2 = \frac{\partial}{\partial x}, \quad Y_3 = x\frac{\partial}{\partial t} + t\frac{\partial}{\partial x},$$

$$Y_4 = t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} - 2f\frac{\partial}{\partial f},$$

$$Y_\varphi = \varphi(x)\frac{\partial}{\partial u} + [\varphi' f + \varphi''(u_t^2 - u_x^2)]\frac{\partial}{\partial f},$$

where $\varphi = \varphi(x)$ is an arbitrary function.

5 Equivalence groups for evolution equations

5.1 Generalised Burgers equation

The generalised Burgers equation

$$u_t + uu_x + f(t)u_{xx} = 0, \tag{5.1}$$

has applications in acoustic phenomena. It has been also used to model turbulence and certain steady state viscous flows. The group $\mathcal{E}$ of equivalence transformations for Eqs. (5.1) was calculated in [71] (see also [63]) by the direct method. The group $\mathcal{E}$ comprises the linear transformation:

$$\bar{x} = c_3c_5x + c_1c_5^2t + c_2,\ \bar{t} = c_5^2t + c_4,\ \bar{u} = \frac{c_3}{c_5}u + c_1 \tag{5.2}$$

and the projective transformation:

$$\bar{x} = \frac{c_3c_6x - c_1}{c_6^2t - c_4} + c_2,\ \bar{t} = c_5 - \frac{1}{c_6^2t - c_4},\ \bar{u} = c_3c_6(ut - x) + \frac{c_3c_4}{c_6}u + c_1, \tag{5.3}$$

where $c_1, \ldots, c_6$ are constants such that $c_3 \neq 0, c_5 \neq 0$, and $c_6 \neq 0$. Under these transformations, the coefficient $f(t)$ of the Eq. (5.1) is mapped to

$$\bar{f} = c_3^2 f. \tag{5.4}$$

It is manifest from Eqs. (5.2) and (5.3) that the continuous equivalence group $\mathcal{E}_c$ is generated by the six-dimensional equivalence algebra $L_{\mathcal{E}}$ spanned by

$$
\begin{aligned}
& Y_1 = \frac{\partial}{\partial t}\,, \ Y_2 = \frac{\partial}{\partial x}\,, \ Y_3 = t\frac{\partial}{\partial x} + \frac{\partial}{\partial u}\,, \ Y_4 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} - u\frac{\partial}{\partial u}\,, \\
& Y_5 = x\frac{\partial}{\partial x} + u\frac{\partial}{\partial u} + 2f\frac{\partial}{\partial f}\,, \quad Y_6 = t^2\frac{\partial}{\partial t} + xt\frac{\partial}{\partial x} + (x - ut)\frac{\partial}{\partial u}\,\cdot
\end{aligned}
\tag{5.5}
$$

5.2 Equations $u_t = u_{xx} + g(x, u, u_x)$

We showed in [65] that this equation has the infinite continuous equivalence group $\mathcal{E}$ with Lie algebra $L_{\mathcal{E}}$ is spanned by the operators

$$
\begin{aligned}
& Y_1 = \frac{\partial}{\partial t}\,, \quad Y_2 = \frac{\partial}{\partial x}\,, \quad Y_3 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} - 2g\frac{\partial}{\partial g}\,, \\
& Y_\psi = \psi(x, u)\frac{\partial}{\partial u} + [g\psi_u - (\psi_{uu}u_x^2 + 2\psi_{ux}u_x + \psi_{xx})]\frac{\partial}{\partial g}\,\cdot
\end{aligned}
\tag{5.6}
$$

5.3 Equations $u_t = f(x, u)u_{xx} + g(x, u, u_x)$

In [64], we considered the equivalence group and calculated the invariants for the family of evolution equations of the form

$$
u_t = f(x, u)u_{xx} + g(x, u, u_x). \tag{5.7}
$$

A number of particular cases of this class of equations have been used to model physical problems. Such examples are the well-known nonlinear diffusion equation

$$
u_t = [D(u)u_x]_x,
$$

and its modifications, e.g. equations of the form

$$
u_t = [g(x)D(u)u_x]_x,
$$

$$
u_t = [g(x)D(u)u_x]_x - K(u)u_x,
$$

$$
u_t = (u^n)_{xx} + g(x)u^m + f(x)u^s u_x.
$$

The generalised Burgers equation (5.1) is also a particular case of Eq. (5.7).

The class of equations (5.7) has an infinite continuous equivalence group $\mathcal{E}_c$ with the infinite-dimensional Lie algebra $L_{\mathcal{E}}$ spanned by the operators

$$Y_1 = \frac{\partial}{\partial t}, \quad Y_2 = t\frac{\partial}{\partial t} - f\frac{\partial}{\partial f} - g\frac{\partial}{\partial g},$$

$$Y_\phi = \phi(x)\frac{\partial}{\partial x} + 2\phi' f\frac{\partial}{\partial f} + \phi'' f u_x\frac{\partial}{\partial g},$$

$$Y_\psi = \psi(x,u)\frac{\partial}{\partial u} + \left[\psi_u g - (\psi_{uu}u_x^2 + 2\psi_{xu}u_x + \psi_{xx})f\right]\frac{\partial}{\partial g}\cdot$$

5.4 Equations $u_t = f(x,u,u_x)u_{xx} + g(x,u,u_x)$

We showed in [65] that this equation has the infinite continuous equivalence group $\mathcal{E}$ with Lie algebra $L_{\mathcal{E}}$ is spanned by the following operators:

$$\begin{aligned} &Y_1 = \frac{\partial}{\partial t}, \quad Y_2 = t\frac{\partial}{\partial t} - f\frac{\partial}{\partial f} - g\frac{\partial}{\partial g}, \\ &Y_\phi = \phi(x,u)\frac{\partial}{\partial x} + 2(\phi_x + u_x\phi_u)f\frac{\partial}{\partial f} \\ &\quad + \left[u_x(\phi_{xx} + 2u_x\phi_{xu} + u_x^2\phi_{uu})f - u_x\phi_u g\right]\frac{\partial}{\partial g}, \\ &Y_\psi = \psi(x,u)\frac{\partial}{\partial u} + [g\psi_u - (\psi_{uu}u_x^2 + 2\psi_{xu}u_x + \psi_{xx})f]\frac{\partial}{\partial g}\cdot \end{aligned} \tag{5.8}$$

5.5 A model from tumour biology

The system of equations

$$\begin{aligned} u_t &= f(u) - (uc_x)_x, \\ c_t &= -g(c,u), \end{aligned} \tag{5.9}$$

where $f(u)$ and $g(c,u)$ are, in general, arbitrary functions, are used in mathematical biology for describing spread of malignant tumour.

The equivalence transformations for Eqs. (5.9) are calculated in [61]. It is shown that the system (5.9) has the six-dimensional equivalence algebra spanned by the following generators:

$$\begin{aligned} &Y_1 = \frac{\partial}{\partial t}, \quad Y_2 = \frac{\partial}{\partial x}, \quad Y_3 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} - 2f\frac{\partial}{\partial f} - 2g\frac{\partial}{\partial g}, \\ &Y_4 = \frac{\partial}{\partial c}, \quad Y_5 = x\frac{\partial}{\partial x} + 2c\frac{\partial}{\partial c} + 2g\frac{\partial}{\partial g}, \quad Y_6 = u\frac{\partial}{\partial u} + f\frac{\partial}{\partial f}. \end{aligned} \tag{5.10}$$

6 Examples from nonlinear acoustics

The equation

$$\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x} - u\frac{\partial u}{\partial t}\right) = -\beta\, u, \quad \beta = \text{const} \neq 0$$

is used in nonlinear acoustics for describing several physical phenomena. The following two generalizations of this model and t heir equivalence groups were given in [59] in accordance with our *principle of a priori use of symmetries.* The first generalized model has the form

$$\frac{\partial}{\partial t}\left[\frac{\partial u}{\partial x} - P(u)\frac{\partial u}{\partial t}\right] = F(x, u). \tag{6.1}$$

Its equivalence algebra $L_{\mathcal{E}}$ is a seven-dimensional Lie algebra spanned by

$$Y_1 = \frac{\partial}{\partial t}\,, \quad Y_2 = \frac{\partial}{\partial x}\,, \quad Y_3 = \frac{\partial}{\partial u}\,, \quad Y_4 = t\frac{\partial}{\partial t} + u\frac{\partial}{\partial u} + P\frac{\partial}{\partial P}\,,$$

$$Y_5 = u\frac{\partial}{\partial u} + F\frac{\partial}{\partial F}\,, \quad Y_6 = x\frac{\partial}{\partial t} - \frac{\partial}{\partial P}\,, \quad Y_7 = x\frac{\partial}{\partial x} - P\frac{\partial}{\partial P} - F\frac{\partial}{\partial F}\,\cdot \tag{6.2}$$

The second generalized model has the form

$$\frac{\partial}{\partial t}\left[\frac{\partial u}{\partial x} - Q(x, u)\frac{\partial u}{\partial t}\right] = F(x, u). \tag{6.3}$$

Its equivalence algebra is infinite-dimensional and comprises the operators

$$Y_1 = \varphi(x)\frac{\partial}{\partial t} - \varphi'(x)\frac{\partial}{\partial Q}\,, \quad Y_2 = \psi(x)\frac{\partial}{\partial x} - \psi'(x)Q\frac{\partial}{\partial Q} - \psi'(x)F\frac{\partial}{\partial F}\,,$$

$$Y_3 = \lambda(x)\frac{\partial}{\partial u}\,, \quad Y_4 = t\frac{\partial}{\partial t} + u\frac{\partial}{\partial u} + Q\frac{\partial}{\partial Q}\,, \quad Y_5 = u\frac{\partial}{\partial u} + F\frac{\partial}{\partial F}\,\cdot \tag{6.4}$$

7 Invariants of linear ODEs

Here, the method of calculation of invariants will be illustrated by the third-order equation (see Section 2.3)

$$y''' + 3c_1(x)y'' + 3c_2(x)y' + c_3(x)y = 0. \tag{7.1}$$

Definition 5.3. A function

$$h = h(x, y, c, c', c'', \ldots) \tag{7.2}$$

of the variables x, y and of the coefficients $c = (c_1, c_2, c_3)$ together with their derivatives $c',\ c'', \ldots$ of a finite order is called an *invariant* of the family of linear equations (7.1) if h is a differential invariant for the equivalence transformations (2.17)-(2.18)*. We call h a *semi-invariant* if it is invariant only under the subgroup comprising the linear transformation (2.17)†. The *order of the invariant* is the highest order of derivatives $c',\ c'', \ldots$ involved in h.

Remark 5.4. The independent variable x is manifestly semi-invariant. Therefore, we can ignore it in calculating semi-invariants.

Theorem 5.9. Equation (7.1) has two independent semi-invariants of the first order:

$$h_1 = c_2 - c_1^2 - c_1', \tag{7.3}$$
$$h_2 = c_3 - 3c_1c_2 + 2c_1^3 + 2c_1c_1' - c_2'. \tag{7.4}$$

Any semi-invariant is a function of x, y and $h_1, h_2, h_1', h_2', \ldots$.

Proof. Firs we check that there are no semi-invariants of the order 0, i.e. of the form

$$h = h(y, c_1, c_2, c_3).$$

We write the invariance test $Y_\eta(h) = 0$ for the equivalence generator (2.28):

$$\eta\frac{\partial h}{\partial y} + \eta'\frac{\partial h}{\partial c_1} + \left(\eta'' + 2c_1\eta'\right)\frac{\partial h}{\partial c_2} + \left(\eta''' + 3c_1\eta'' + 3c_2\eta'\right)\frac{\partial h}{\partial c_3} = 0. \tag{7.5}$$

Since the function $\eta(x)$ is arbitrary, and hence there are no relations between its derivatives, Eq. (7.5) splits into four equations obtained by annulling separately the terms with η, η''', η'' and η' :

$$\frac{\partial h}{\partial y} = 0, \quad \frac{\partial h}{\partial c_3} = 0, \quad \frac{\partial h}{\partial c_2} + 3c_1\frac{\partial h}{\partial c_3} = 0, \quad \frac{\partial h}{\partial c_1} + 2c_1\frac{\partial h}{\partial c_2} + 3c_2\frac{\partial h}{\partial c_3} = 0.$$

*It means that h is invariant under the equivalence group prolonged to the derivatives $c',\ c'', \ldots$.

†One can consider other semi-invariants by taking, instead of (2.17), any subgroups of the general equivalence group. Note, that the classical papers [12], [72], [73], [86], [24], [25], [20] mentioned in Introduction deal exclusively with invariants of the subgroup (2.17).

It follows that $h = \text{const.}$, i.e. there are no differential invariants of the order 0.

Now we take the first prolongation of Y_η and solve the equation

$$\eta\frac{\partial h}{\partial y} + \eta'\frac{\partial h}{\partial c_1} + \left[\eta'' + 2c_1\eta'\right]\frac{\partial h}{\partial c_2} + \left[\eta''' + 3c_1\eta'' + 3c_2\eta'\right]\frac{\partial h}{\partial c_3} + \eta''\frac{\partial h}{\partial c_1'}$$

$$+\left[\eta''' + 2c_1\eta'' + 2c_1'\eta'\right]\frac{\partial h}{\partial c_2'} + \left[\eta^{(iv)} + 3c_1\eta''' + 3c_2\eta'' + 3c_1'\eta'' + 3c_2'\eta'\right]\frac{\partial h}{\partial c_3'} = 0$$

by letting

$$h = h(y, c_1, c_2, c_3, c_1', c_2', c_3').$$

Again, the term with η yields that h does nod depend upon y, whereas the term with $\eta^{(iv)}$ yields $\partial h/\partial c_3' = 0$. The terms with η''', η'', η' give three linear partial differential equations for the function $h = h(c_1, c_2, c_3, c_1', c_2')$. These equations have precisely two functionally independent solutions given in (7.3) - (7.4).

We have to continue by taking the second prolongation of Y_η and considering the second-order semi-invariants, i.e. letting $h = h(c, c', c'')$. However, one can verify that the equation $Y_\eta(h) = 0$, where Y_η is the twice-prolonged operator, has precisely four functionally independent solutions. Since h_1 and h_2 together with their first derivatives provide four functionally independent solutions of this type, the theorem is proved for semi-invariants of the second order. The iteration completes the proof.

Remark 5.5. The semi-invariants for third-order equations were calculated by Laguerre (see [73]). He found the first-order semi-invariant (7.3) and a second-order one, $\widetilde{h}_2 = c_3 - 3c_1c_2 + 2c_1^3 - c_1''$, instead of (7.3). They are equivalent, namely, $\widetilde{h}_2 = h_2 + h_1'$.

Let us turn now to the proper invariants. The infinitesimal transformation (2.18),

$$\overline{x} \approx x + \varepsilon\xi(x),$$

implies the following infinitesimal transformations of derivatives:

$$\begin{aligned} y' &\approx (1 + \varepsilon\xi')\overline{y}', \\ y'' &\approx (1 + 2\varepsilon\xi')\overline{y}'' + \varepsilon\overline{y}'\xi'', \\ y''' &\approx (1 + 3\varepsilon\xi')\overline{y}''' + 3\varepsilon\overline{y}''\xi'' + \varepsilon\overline{y}'\xi'''. \end{aligned}$$

Consequently Eq. (7.1) becomes

$$\overline{y}''' + 3\overline{c}_1\overline{y}'' + 3\overline{c}_2\overline{y}' + \overline{c}_3\overline{y} = 0,$$

where

$$\bar{c}_1 \approx c_1 + \varepsilon(\xi'' - c_1\xi'),$$
$$\bar{c}_2 \approx c_2 + \varepsilon\Big(\frac{1}{3}\xi''' + c_1\xi'' - 2c_2\xi'\Big),$$
$$\bar{c}_3 \approx c_3 - 3\varepsilon c_3\xi'.$$

The corresponding group generator, extended to the first derivatives of c_1, c_2 is

$$Y_\xi = \xi\frac{\partial}{\partial x} + \big(\xi'' - c_1\xi'\big)\frac{\partial}{\partial c_1} + \Big(\frac{1}{3}\xi''' + c_1\xi'' - 2c_2\xi'\Big)\frac{\partial}{\partial c_2} - 3c_3\xi'\frac{\partial}{\partial c_3}$$
$$+ \big(\xi''' - c_1\xi'' - 2c_1'\xi'\big)\frac{\partial}{\partial c_1'} + \Big(\frac{1}{3}\xi^{(iv)} + c_1\xi''' + c_1'\xi'' - 2c_2\xi'' - 3c_2'\xi'\Big)\frac{\partial}{\partial c_2'}\,\cdot$$

We rewrite it in terms of the semi-invariants (7.3)- (7.4) and after prolongation obtain:

$$Y_\xi = \xi\frac{\partial}{\partial x} - \Big[\frac{2}{3}\xi''' + 2h_1\xi'\Big]\frac{\partial}{\partial h_1} - \Big[\frac{1}{3}\xi^{(iv)} + h_1\xi'' + 3h_2\xi'\Big]\frac{\partial}{\partial h_2}$$
$$- \Big[\frac{2}{3}\xi^{(iv)} + 2h_1\xi'' + 3h_1'\xi'\Big]\frac{\partial}{\partial h_1'} - \Big[\frac{2}{3}\xi^{(v)} + 2h_1\xi''' + 5h_1'\xi'' + 4h_1''\xi'\Big]\frac{\partial}{\partial h_1''}$$
$$- \Big[\frac{1}{3}\xi^{(v)} + h_1\xi''' + h_1'\xi'' + 3h_2\xi'' + 4h_2'\xi'\Big]\frac{\partial}{\partial h_2'} + \ldots\,. \tag{7.6}$$

The following results were obtained in [38] (see also [39], Section 10.2) by applying to the operator (7.6) the approach used in the proof of Theorem 5.9.

Theorem 5.10. Eq. (7.1) has a singular invariant equation with respect to the group of general equivalence transformations, namely, the equation

$$h_1' - 2h_2 = 0, \tag{7.7}$$

where h_1 and h_2 are the semi-invariants (7.3) and (7.4), respectively.

Theorem 5.11. The *least invariant* of equation (7.1), i.e. an invariant involving the derivatives of h_1 and h_2 of the lowest order is

$$\theta = \frac{1}{\lambda^2}\left[7\left(\frac{\lambda'}{\lambda}\right)^2 - 6\frac{\lambda''}{\lambda} + 27h_1\right]^3, \tag{7.8}$$

where

$$\lambda = h_1' - 2h_2. \tag{7.9}$$

The higher-order invariants are obtained from θ by means of invariant differentiation, and any invariant of an arbitrary order is a function of θ and its invariant derivatives.

Corollary 5.1. Eq. (7.1) is equivalent to the equation

$$y''' = 0$$

if and only if $\lambda = 0$, i.e. the invariant equation (7.7) holds (see [73]).

Corollary 5.2. The necessary and sufficient condition for Eq. (7.1) to be equivalent to

$$y''' + y = 0$$

is that $\lambda \neq 0$ and that $\theta = 0$. For example, the equation

$$y''' + c(x)y = 0$$

is equivalent to

$$z''' + z = 0$$

only in the case

$$c(x) = (kx + l)^{-6},$$

where the constants k and l do not vanish simultaneously.

8 Invariants of hyperbolic second-order linear PDEs

In this section and sections 9, 10 we will discuss the invariants for all three types of equations, hyperbolic, elliptic and parabolic, with two independent variables. The calculations are based on my recent works [41], [43] and illustrate the application of our method to partial differential equations with infinite equivalence groups.

8.1 Equivalence transformations

Consider the general hyperbolic equation written in the characteristic variables x, y, i.e. in the following standard form:

$$u_{xy} + a(x,y)u_x + b(x,y)u_y + c(x,y)u = 0. \tag{8.1}$$

Recall that an *equivalence transformation* of equations (8.1) is defined as an invertible transformation

$$\overline{x} = f(x,y,u), \quad \overline{y} = g(x,y,u), \quad \overline{u} = h(x,y,u) \tag{8.2}$$

such that the equation (8.1) with any coefficients a, b, c remains linear and homogeneous but the transformed equation can have, in general, new coefficients $\overline{a}, \overline{b}, \overline{c}$. Two equations of the form (8.1) are are said to be *equivalent* if they can be connected by a properly chosen equivalence transformation. Proceeding as in Sections 3.1 and 4.1, we prove the following result.

Theorem 5.12. The equivalence algebra $L_{\mathcal{E}}$ for Eqs. (8.1) comprises the operators

$$Y = \xi(x)\frac{\partial}{\partial x} + \eta(y)\frac{\partial}{\partial y} + \sigma(x,y)\, u\frac{\partial}{\partial u} + \mu^1\frac{\partial}{\partial a} + \mu^2\frac{\partial}{\partial b} + \mu^3\frac{\partial}{\partial c}\,, \tag{8.3}$$

where $\xi(x), \eta(y), \sigma(x,y)$ are arbitrary functions, and μ^i are given by

$$\begin{aligned}
\mu^1 &= -(\sigma_y + a\eta'),\\
\mu^2 &= -(\sigma_x + b\xi'),\\
\mu^3 &= -(\sigma_{xy} + a\,\sigma_x + b\,\sigma_y + c\,\eta' + c\,\xi').
\end{aligned} \tag{8.4}$$

The operator (8.3) generates the continuous infinite group $\mathcal{E}_c$ of equivalence transformations (8.2) composed of the linear transformation of the dependent variable:

$$\overline{u} = \phi(x,y)\,u, \quad \phi(x,y) \neq 0, \tag{8.5}$$

and invertible changes of the independent variables of the form:

$$\overline{x} = f(x), \quad \overline{y} = g(y), \tag{8.6}$$

where $\phi(x,y), f(x)$ and $g(y)$ are arbitrary functions such that $f'(x) \neq 0, g'(y) \neq 0$.

Remark 5.6. The general group $\mathcal{E}$ of equivalence transformations (8.2) for the family of hyperbolic equations (8.1) consists of the continuous group $\mathcal{E}_c$ augmented by the interchange of the variables,

$$x_1 = y, \quad y_1 = x. \tag{8.7}$$

Hence, the group $\mathcal{E}$ contains, along with (8.6), the change of variables

$$\tilde{x} = r(y), \quad \tilde{y} = s(x) \tag{8.8}$$

obtained by taking the composition of (8.6) and (8.7).

8.2 Semi-invariants

In what follows, we will use only the continuous equivalence group of transformations (8.5)-(8.5) written in the form:

$$u = \varphi(x, y)\, v, \quad \varphi(x, y) \neq 0, \tag{8.9}$$

$$\overline{x} = f(x), \quad \overline{y} = g(y), \tag{8.10}$$

where $v = v(\bar{x}, \bar{y})$ is a new dependent variable.

Definition 5.4. A function

$$J = J(x, y, a, b, c, a_x, a_y, \ldots) \tag{8.11}$$

is called an *invariant* of the family of hyperbolic equations (8.1) if it is a differential invariant for the equivalence group (8.5)-(8.5). We call J a *semi-invariant* if it is invariant only under the linear transformation (8.9) of the dependent variable.

Let us find all semi-invariants. The apparent semi-invariants x and y are not considered in further calculations. One can proceed by using directly the operator (8.3) by letting $\xi = \eta = 0$, but one does not need to remember the expressions (8.4) for the coefficients μ. Indeed, we consider the infinitesimal transformation (8.9) by letting

$$\varphi(x, y) \approx 1 + \varepsilon\sigma(x, y),$$

where ε is a small parameter. Thus, we have:

$$u \approx [1 + \varepsilon\sigma(x, y)]v. \tag{8.12}$$

The transformation of derivatives is written, in the first order of precision in ε, as follows:

$$u_x \approx (1 + \varepsilon\sigma)v_x + \varepsilon\sigma_x v, \quad u_y \approx (1 + \varepsilon\sigma)v_y + \varepsilon\sigma_y v,$$

$$u_{xy} \approx (1 + \varepsilon\sigma)v_{xy} + \varepsilon\sigma_y v_x + \varepsilon\sigma_x v_y + \varepsilon\sigma_{xy} v. \tag{8.13}$$

Therefore,

$$u_{xy} + au_x + bu_y + cu \approx (1+\varepsilon\sigma)v_{xy} + \varepsilon\sigma_y v_x + \varepsilon\sigma_x v_y + \varepsilon\sigma_{xy} v$$

$$+(1+\varepsilon\sigma)av_x + \varepsilon\sigma_x av + (1+\varepsilon\sigma)bv_y + \varepsilon\sigma_y bv(1+\varepsilon\sigma)cv,$$

whence the infinitesimal transformation of the equation (8.1):

$$v_{xy} + (a+\varepsilon\sigma_y)\, v_x + (b+\varepsilon\sigma_x)\, v_y + [c+\varepsilon(\sigma_{xy} + a\sigma_x + b\sigma_y)]\, v = 0.$$

Thus, the coefficients of the equation (8.1) undergo the infinitesimal transformations

$$\overline{a} \approx a + \varepsilon\sigma_y, \quad \overline{b} \approx b + \varepsilon\sigma_x, \quad \overline{c} \approx c + \varepsilon(\sigma_{xy} + a\sigma_x + b\sigma_y), \tag{8.14}$$

that provide the generator (cf. 8.3))

$$Z = \sigma_y \frac{\partial}{\partial a} + \sigma_x \frac{\partial}{\partial b} + \Big(\sigma_{xy} + a\sigma_x + b\sigma_y\Big) \frac{\partial}{\partial c} \cdot \tag{8.15}$$

Let us first consider the functions (8.11) of the form $J = J(a,b,c)$. Then the infinitesimal invariant test $Z(J) = 0$ is written:

$$\sigma_y \frac{\partial J}{\partial a} + \sigma_x \frac{\partial J}{\partial b} + \Big(\sigma_{xy} + a\sigma_x + b\sigma_y\Big) \frac{\partial J}{\partial c} = 0.$$

Since the function $\sigma(x,y)$ is arbitrary, the latter equation splits into the following three equations obtained by annulling separately the terms with σ_{xy}, σ_x and σ_y :

$$\frac{\partial J}{\partial c} = 0, \quad \frac{\partial J}{\partial b} = 0, \quad \frac{\partial J}{\partial a} = 0.$$

Thus, there are no invariants $J(a,b,c)$ other than $J = \text{const}$.

Therefore, one should consider, as the next step, the semi-invariants involving first-order derivatives of the coefficients a, b, c, i.e. the (8.11) of the form

$$J = J(a, b, c, a_x, a_y, b_x, b_y, c_x, c_y).$$

Accordingly, we take the first prolongation of the generator (8.15):

$$Z = \sigma_y \frac{\partial}{\partial a} + \sigma_x \frac{\partial}{\partial b} + \Big(\sigma_{xy} + a\sigma_x + b\sigma_y\Big) \frac{\partial}{\partial c} + \sigma_{xy} \frac{\partial}{\partial a_x} + \sigma_{yy} \frac{\partial}{\partial a_y} + \sigma_{xx} \frac{\partial}{\partial b_x} + \sigma_{xy} \frac{\partial}{\partial b_y}$$

$$+ \Big(\sigma_{xxy} + a\sigma_{xx} + a_x\sigma_x + b\sigma_{xy} + b_x\sigma_y\Big) \frac{\partial}{\partial c_x} + \Big(\sigma_{xyy} + a\sigma_{xy} + a_y\sigma_x + b\sigma_{yy} + b_y\sigma_y\Big) \frac{\partial}{\partial c_y} \cdot$$

The equation $Z(J) = 0$, upon equating to zero at first the terms with $\sigma_{xxy}, \sigma_{xyy}$ and then with σ_{xx}, σ_{yy}, yields $\partial J/\partial c_x = 0, \partial J/\partial c_y = 0$ and

$\partial J/\partial b_x = 0, \partial J/\partial a_y = 0$, respectively. Hence, $J = J(a, b, c, a_x, b_y)$. Now the terms with σ_{xy}, σ_x and σ_y provide the following system of three equations:

$$\frac{\partial J}{\partial c} + \frac{\partial J}{\partial a_x} + \frac{\partial J}{\partial b_y} = 0, \quad \frac{\partial J}{\partial b} + a\frac{\partial J}{\partial c} = 0, \quad \frac{\partial J}{\partial a} + b\frac{\partial J}{\partial c} = 0.$$

One can readily solve the last two equations of this system to obtain $J = J(\lambda, a_x, b_y)$, where $\lambda = ab - c$. Then the first equation of the system yields:

$$\frac{\partial J}{\partial a_x} + \frac{\partial J}{\partial b_y} - \frac{\partial J}{\partial \lambda} = 0.$$

The latter equation has two functionally independent solutions, e.g.

$$J_1 = a_x - b_y \quad J_2 = a_x + \lambda \equiv a_x + ab - c. \tag{8.16}$$

Denoting $h = J_2$ and $k = J_2 - J_1$, one obtains two independent *semi-invariants* of the equation (8.1), namely, the *Laplace invariants:*

$$h = a_x + ab - c, \qquad k = b_y + ab - c. \tag{8.17}$$

One can verify, by considering the higher-order prolongations, that the semi-invariants involving higher-order derivatives of a, b, c are obtained merely by differentiating the Laplace invariants (8.17). Namely, we have the following theorem.

Theorem 5.13. The general semi-invariant for Eqs. (8.1) has the following form:

$$J = J(x, y, h, k, h_x, h_y, k_x, k_y, h_{xx}, h_{xy}, h_{yy}, k_{xx}, k_{xy}, k_{yy}, \ldots). \tag{8.18}$$

Proof. Let us consider the second-order semi-invariants. The differentiations of the Laplace invariants with respect to x and y provide four second-order semi-invariants h_x, h_y, k_x, and h_y. On the other hand, the general form of second-order semi-invariants involves *nine* additional variables, namely,

$$a_{xx},\ a_{xy},\ a_{yy},\ b_{xx},\ b_{xy},\ b_{yy},\ c_{xx},\ c_{xy},\ c_{yy}.$$

However, the invariance condition with respect to the second prolongation of the generator (8.15) brings *five* additional equations due to the five derivatives of $\sigma(x, y)$, namely (see the first prolongation of the generator (8.15))

$$\sigma_{xxxy}, \quad \sigma_{xxyy}, \quad \sigma_{xyyy}, \quad \sigma_{xxx}, \quad \sigma_{yyy}.$$

These additional equations reduce by five the number of the additional variables involved in second-order semi-invariants. In consequence, we have precisely *four* independent second-order semi-invariants, namely h_x, h_y, k_x, and h_y. The similar reasoning for the higher-order prolongations of the generator (8.15) completes the proof.

8.3 Laplace's problem. Calculation of invariants

Laplace's problem: Find all invariants for the family of the hyperbolic equations (8.1).

Here we will discuss the solution of Laplace's problem given in [43]. An arbitrary *invariant* of equation (8.1) is obtained by subjecting the general semi-invariant (8.18) to the invariance test under the changes (8.10) of the independent variables.

The infinitesimal transformation (8.10) of the variable x has the form

$$\overline{x} \approx x + \varepsilon\xi(x) \tag{8.19}$$

and yields:

$$u_x \approx (1+\varepsilon\xi')u_{\bar{x}}, \quad u_y = u_{\bar{y}}, \quad u_{xy} \approx (1+\varepsilon\xi')u_{\bar{x}\bar{y}},$$

where $\xi' = d\xi(x)/dx$. Hence, equation (8.1) undergoes the infinitesimal transformation

$$(1+\varepsilon\xi')u_{\bar{x}\bar{y}} + a(1+\varepsilon\xi')u_{\bar{x}} + bu_{\bar{y}} + cu = 0,$$

and can be written, in the first order of precision in ε, in the form (8.1):

$$u_{\bar{x}\bar{y}} + au_{\bar{x}} + (b - \varepsilon\xi' b)u_{\bar{y}} + (c - \varepsilon\xi' c)u = 0.$$

It provides the infinitesimal transformation of the coefficients of equation (8.1):

$$\overline{a} \approx a, \quad \overline{b} \approx b - \varepsilon\xi' b, \quad \overline{c} \approx c - \varepsilon\xi' c. \tag{8.20}$$

The infinitesimal transformations (8.19) and (8.20) define the generator (cf. 8.3))

$$X = -\xi(x)\frac{\partial}{\partial x} + \xi' b\frac{\partial}{\partial b} + \xi' c\frac{\partial}{\partial c}\,\cdot \tag{8.21}$$

The prolongation of the generator (8.21) to a_x and b_y has the form

$$X = -\xi(x)\frac{\partial}{\partial x} + \xi'(x)\Big[b\frac{\partial}{\partial b} + c\frac{\partial}{\partial c} + a_x\frac{\partial}{\partial a_x} + b_y\frac{\partial}{\partial b_y}\Big]$$

and furnishes the following action on Laplace's invariants:

$$X = -\xi(x)\frac{\partial}{\partial x} + \xi'(x)\Big[h\,\frac{\partial}{\partial h} + k\,\frac{\partial}{\partial k}\Big]\cdot \tag{8.22}$$

Now we will look for the invariants (8.18) involving the derivatives of h and k up to second order. Therefore, we apply the usual prolongation procedure and obtain the following second-order prolongation of the operator (8.22):

$$X = -\xi(x)\frac{\partial}{\partial x} + \xi' h\,\frac{\partial}{\partial h} + \xi' k\,\frac{\partial}{\partial k} + (\xi'' h\, + 2\xi' h_x)\frac{\partial}{\partial h_x} + (\xi'' k\, + 2\xi' k_x)\frac{\partial}{\partial k_x}$$

$$+\xi' h_y\,\frac{\partial}{\partial h_y} + \xi' k_y\,\frac{\partial}{\partial k_y} + (\xi''' h\, + 3\xi'' h_x + 3\xi' h_{xx})\frac{\partial}{\partial h_{xx}} + (\xi'' h_y\, + 2\xi' h_{xy})\frac{\partial}{\partial h_{xy}}$$

$$+\xi' h_{yy}\,\frac{\partial}{\partial h_{yy}} + (\xi''' k + 3\xi'' k_x + 3\xi' k_{xx})\frac{\partial}{\partial k_{xx}} + (\xi'' k_y + 2\xi' k_{xy})\frac{\partial}{\partial k_{xy}} + \xi' k_{yy}\,\frac{\partial}{\partial k_{yy}}\,\cdot$$

Since the function $\xi(x)$ is arbitrary, its derivatives $\xi'(x), \xi''(x), \xi'''(x)$ can be treated as new arbitrary functions. Consequently, singling out in the above operator the terms with different derivatives of $\xi(x)$, one obtains the following independent generators:

$$X_\xi = \frac{\partial}{\partial x}\,, \qquad X_{\xi'''} = h\,\frac{\partial}{\partial h_{xx}} + k\,\frac{\partial}{\partial k_{xx}}\,, \tag{8.23}$$

$$X_{\xi'} = h\,\frac{\partial}{\partial h} + k\,\frac{\partial}{\partial k} + 2h_x\,\frac{\partial}{\partial h_x} + h_y\,\frac{\partial}{\partial h_y} + 2k_x\,\frac{\partial}{\partial k_x} + k_y\,\frac{\partial}{\partial k_y} + 3h_{xx}\,\frac{\partial}{\partial h_{xx}}$$

$$+ 2h_{xy}\,\frac{\partial}{\partial h_{xy}} + h_{yy}\,\frac{\partial}{\partial h_{yy}} + 3k_{xx}\,\frac{\partial}{\partial k_{xx}} + 2k_{xy}\,\frac{\partial}{\partial k_{xy}} + k_{yy}\,\frac{\partial}{\partial k_{yy}}\,,$$

$$X_{\xi''} = h\,\frac{\partial}{\partial h_x} + k\,\frac{\partial}{\partial k_x} + 3h_x\,\frac{\partial}{\partial h_{xx}} + h_y\,\frac{\partial}{\partial h_{xy}} + 3k_x\,\frac{\partial}{\partial k_{xx}} + k_y\,\frac{\partial}{\partial k_{xy}}\,\cdot$$

Likewise, the infinitesimal transformation (8.10) of the variable y,

$$\overline{y} \approx y + \varepsilon\eta(y) \tag{8.24}$$

provides the following generator:

$$Y = -\eta(y)\frac{\partial}{\partial y} + \eta'(y)\Big[h\,\frac{\partial}{\partial h} + k\,\frac{\partial}{\partial k}\Big]\,\cdot \tag{8.25}$$

Its second prolongation gives rise to the following independent generators:

$$Y_\eta = \frac{\partial}{\partial y}\,, \qquad Y_{\eta'''} = h\,\frac{\partial}{\partial h_{yy}} + k\,\frac{\partial}{\partial k_{yy}}\,, \tag{8.26}$$

$$Y_{\eta'} = h\,\frac{\partial}{\partial h} + k\,\frac{\partial}{\partial k} + h_x\,\frac{\partial}{\partial h_x} + 2h_y\,\frac{\partial}{\partial h_y} + k_x\,\frac{\partial}{\partial k_x} + 2k_y\,\frac{\partial}{\partial k_y} + h_{xx}\,\frac{\partial}{\partial h_{xx}}$$

$$+ 2h_{xy}\,\frac{\partial}{\partial h_{xy}} + 3h_{yy}\,\frac{\partial}{\partial h_{yy}} + k_{xx}\,\frac{\partial}{\partial k_{xx}} + 2k_{xy}\,\frac{\partial}{\partial k_{xy}} + 3k_{yy}\,\frac{\partial}{\partial k_{yy}}\,,$$

$$Y_{\eta''} = h\,\frac{\partial}{\partial h_y} + k\,\frac{\partial}{\partial k_y} + h_x\,\frac{\partial}{\partial h_{xy}} + 3h_y\,\frac{\partial}{\partial h_{yy}} + k_x\,\frac{\partial}{\partial k_{xy}} + 3k_y\,\frac{\partial}{\partial k_{yy}}\,\cdot$$

It follows from the the invariance condition under the translations, i.e. from the equations $X_\xi(J) = 0$ and $Y_\eta(J) = 0$, that J in (8.18) does not depend upon x and y. It is also evident from (8.26), that the equations

$$h = 0, \quad k = 0 \tag{8.27}$$

are invariant under the operators (8.26). In what follows, we assume that the Laplace invariants do not vanish simultaneously, e.g. $h \neq 0$. The equation $X_{\xi'} J = 0$ for $J = J(h, k)$ gives one of Ovsyannikov's invariants [90], namely

$$p = \frac{k}{h}\,\cdot \tag{8.28}$$

One can readily verify by inspection that p satisfies the invariance test for all operators (8.23) and (8.26). Moreover, the equations $X_{\xi'''}(J) = 0$ and $Y_{\eta'''}(J) = 0$ show that h_{xx}, h_{yy}, k_{xx}, and k_{yy} can appear only in the combinations

$$r = k_{xx} - p\,h_{xx}, \qquad s = k_{yy} - p\,h_{yy}. \tag{8.29}$$

Thus, the general form (8.18) for the second-order invariants is reduced to

$$J(h, p, h_x, h_y, k_x, k_y, h_{xy}, k_{xy}, r, s). \tag{8.30}$$

One has to subject the function (8.30) to the invariance test

$$X_{\xi'}(J) = 0, \quad X_{\xi''}(J) = 0, \quad Y_{\eta'}(J) = 0, \quad Y_{\eta''}(J) = 0, \tag{8.31}$$

where the operators $X_{\xi'}, X_{\xi''}, Y_{\eta'}$, and $Y_{\eta''}$ are rewritten in terms of the variables involved in (8.30) and have the form:

$$\begin{aligned}
X_{\xi'} = \;& h\,\frac{\partial}{\partial h} + 2h_x\,\frac{\partial}{\partial h_x} + h_y\,\frac{\partial}{\partial h_y} + 2k_x\,\frac{\partial}{\partial k_x} + k_y\,\frac{\partial}{\partial k_y}\\
& +2h_{xy}\,\frac{\partial}{\partial h_{xy}} + 2k_{xy}\,\frac{\partial}{\partial k_{xy}} + 3r\,\frac{\partial}{\partial r} + s\frac{\partial}{\partial s}\,,\\
X_{\xi''} = \;& h\,\frac{\partial}{\partial h_x} + p\,h\,\frac{\partial}{\partial k_x} + h_y\,\frac{\partial}{\partial h_{xy}} + k_y\,\frac{\partial}{\partial k_{xy}} + 3(k_x - p\,h_x)\frac{\partial}{\partial r}\,,\\
Y_{\eta'} = \;& h\,\frac{\partial}{\partial h} + h_x\,\frac{\partial}{\partial h_x} + 2h_y\,\frac{\partial}{\partial h_y} + k_x\,\frac{\partial}{\partial k_x} + 2k_y\,\frac{\partial}{\partial k_y}\\
& +2h_{xy}\,\frac{\partial}{\partial h_{xy}} + 2k_{xy}\,\frac{\partial}{\partial k_{xy}} + r\,\frac{\partial}{\partial r} + 3s\frac{\partial}{\partial s}\,,\\
Y_{\eta''} = \;& h\,\frac{\partial}{\partial h_y} + p\,h\,\frac{\partial}{\partial k_y} + h_x\,\frac{\partial}{\partial h_{xy}} + k_x\,\frac{\partial}{\partial k_{xy}} + 3(k_y - p\,h_y)\frac{\partial}{\partial s}\,\cdot
\end{aligned}\tag{8.32}$$

The operators (8.32) obey the commutator relations

$$[X_{\xi'}, X_{\xi''}] = -X_{\xi''}, \quad [X_{\xi'}, Y_{\eta'}] = 0, \quad [X_{\xi'}, Y_{\eta''}] = 0,$$

$$[X_{\xi''}, Y_{\eta'}] = 0, \quad [X_{\xi''}, Y_{\eta''}] = 0, \quad [Y_{\eta'}, Y_{\eta''}] = -Y_{\eta''},$$

and hence span a four-dimensional Lie algebra.

According to the above table of commutators, it is advantageous to begin the solutions of the system (8.31) with the equations (see [39], Section 4.5.3)

$$\begin{aligned}
X_{\xi''}(J) = \;& h\,\frac{\partial J}{\partial h_x} + p\,h\,\frac{\partial J}{\partial k_x} + h_y\,\frac{\partial J}{\partial h_{xy}} + k_y\,\frac{\partial J}{\partial k_{xy}} + 3(k_x - p\,h_x)\frac{\partial J}{\partial r} = 0,\\
Y_{\eta''}(J) = \;& h\,\frac{\partial J}{\partial h_y} + p\,h\,\frac{\partial J}{\partial k_y} + h_x\,\frac{\partial J}{\partial h_{xy}} + k_x\,\frac{\partial J}{\partial k_{xy}} + 3(k_y - p\,h_y)\frac{\partial J}{\partial s} = 0.
\end{aligned}$$

Integration of the characteristic system for the first equation:

$$\frac{dh_x}{h} = \frac{dk_x}{ph} = \frac{dh_{xy}}{h_y} = \frac{dk_{xy}}{k_y} = \frac{dr}{3(k_x - ph_x)}$$

yields that J involves the variables h, p, h_y, k_y, s and the following combinations:

$$\lambda = k_x - ph_x, \quad \tau = hh_{xy} - h_x h_y, \quad \nu = phk_{xy} - k_x k_y, \quad \omega = hr - 3\lambda h_x.$$

Then the second equation reduces to the form

$$Y_{\eta''}(J) = h\,\frac{\partial J}{\partial h_y} + p\,h\,\frac{\partial J}{\partial k_y} + 3(k_y - p\,h_y)\frac{\partial J}{\partial s} = 0.$$

Integration of this equation shows that $J = J(h, p, \lambda, \mu, \tau, \nu, \omega, \rho)$, where

$$
\begin{aligned}
&\lambda = k_x - ph_x,\ \ \mu = k_y - ph_y,\ \ \tau = hh_{xy} - h_x h_y,\\
&\nu = phk_{xy} - k_x k_y, \quad \omega = hr - 3\lambda h_x, \quad \rho = hs - 3\mu h_y.
\end{aligned}
\tag{8.33}
$$

We solve the equation $(X_{\xi'} - Y_{\eta'})(J) = 0$ written in the variables $h, p, \lambda, \mu, \tau, \nu, \omega, \rho$:

$$
(X_{\xi'} - Y_{\eta'})(J) = \lambda\frac{\partial J}{\partial \lambda} - \mu\frac{\partial J}{\partial \mu} + 2\omega\frac{\partial J}{\partial \omega} - 2\rho\frac{\partial J}{\partial \rho} = 0,
$$

and see that $J = J(h, p, m, \tau, \nu, n, N)$, where

$$
m = \lambda\,\mu, \qquad n = \omega\,\rho, \qquad N = \frac{\omega}{\lambda^2}\,\cdot
\tag{8.34}
$$

To complete integration of the system (8.31) we solve the equation $X_{\xi'}(J) = 0$:

$$
X_{\xi'}(J) = h\frac{\partial J}{\partial h} + 3\tau\frac{\partial J}{\partial \tau} + 3\nu\frac{\partial J}{\partial \nu} + 3m\frac{\partial J}{\partial m} + 6n\frac{\partial J}{\partial n} = 0,
$$

and obtain the following six independent second-order invariants:

$$
p = \frac{k}{h}\,, \quad q_1 = \frac{\tau}{h^3}\,, \quad Q = \frac{\nu}{h^3}\,, \quad N = \frac{\omega}{\lambda^2}\,, \quad M = \frac{n}{h^6}\,, \quad I = \frac{m}{h^3}\,,
\tag{8.35}
$$

provided that $h \neq 0$ and $\lambda \neq 0$. Note, that each of the equations

$$
\lambda \equiv k_x - ph_x = 0, \quad \mu \equiv k_y - ph_y = 0
\tag{8.36}
$$

is invariant. We exclude this case, as well as (8.27), in our calculations.

Let us rewrite the invariants (8.35) in terms of Laplace's semi-invariants h and k, and Ovsyannikov's invariant $p = k/h$. Using the equations

$$
k_x - ph_x \equiv \frac{h\,k_x - k\,h_x}{h} = h\,p_x, \quad k_y - ph_y \equiv \frac{h\,k_y - k\,h_y}{h} = h\,p_y,
$$

we have:

$$
\begin{aligned}
&\lambda = k_x - ph_x = h\,p_x, \qquad \mu = k_y - ph_y = h\,p_y,\\
&r = k_{xx} - p\,h_{xx} = hp_{xx} + 2h_x p_x, \quad \omega = h^2 p_{xx} - hh_x p_x,\\
&s = k_{yy} - p\,h_{yy} = hp_{yy} + 2h_y p_y, \quad \rho = h^2 p_{yy} - hh_y p_y
\end{aligned}
\tag{8.37}
$$

Using these equations, one can easily see that

$$
q_1 = \frac{\tau}{h^3} = \frac{h_{xy}}{h^2} - \frac{h_x h_y}{h^3} \equiv \frac{1}{h}\,\frac{\partial^2 \ln|h|}{\partial x \partial y}\,,
\tag{8.38}
$$

and that $Q = p^3\, q_2$, where q_2 is an invariant (since p^3 is an invariant) defined by

$$q_2 = \frac{1}{k}\,\frac{\partial^2 \ln|k|}{\partial x \partial y}\,\cdot \tag{8.39}$$

Furthermore, we can replace the invariant

$$M = \frac{\omega}{h^6} = \left(\frac{p_x}{h}\right)_x \left(\frac{p_y}{h}\right)_y \tag{8.40}$$

by the invariant

$$H = \frac{\rho}{\mu^2}\,, \tag{8.41}$$

using the equations (8.34)-(8.37) and noting that $M = NHI^2$. Indeed,

$$NH = \frac{\omega\,\rho}{\lambda^2\mu^2} = \frac{\omega\,\rho}{h^4 p_x^2 p_y^2} = \frac{\omega\,\rho}{h^6 I^2}\,\cdot$$

Invoking the definitions (8.33)-(8.35) and the equations (8.37), we have:

$$N = \frac{\omega}{\lambda^2} = \frac{h(k_{xx} - p h_{xx})}{(k_x - p h_x)^2} - \frac{3\,h_x}{k_x - p h_x} = \frac{p_{xx}}{p_x^2} - \frac{h_x}{h p_x} = \frac{1}{p_x}\left(\ln\left|\frac{p_x}{h}\right|\right)_x \cdot \tag{8.42}$$

Likewise, we rewrite the invariant (8.41) in the form

$$H = \frac{\rho}{\mu^2} = \frac{p_{yy}}{p_y^2} - \frac{h_y}{h p_y} = \frac{1}{p_y}\left(\ln\left|\frac{p_y}{h}\right|\right)_y \cdot \tag{8.43}$$

Finally, we have

$$I = \frac{\lambda\,\mu}{h^3} = \frac{p_x\, p_y}{h}\,\cdot \tag{8.44}$$

Collecting together the invariants (8.28), (8.38), (8.39), (8.42), (8.43) and (8.44), we ultimately arrive at the following complete set of invariants of the second order for equation (8.1):

$$p = \frac{k}{h}\,, \quad q_1 = \frac{1}{h}\,\frac{\partial^2 \ln|h|}{\partial x \partial y}\,, \quad q_2 = \frac{1}{k}\,\frac{\partial^2 \ln|k|}{\partial x \partial y}\,, \tag{8.45}$$

$$N = \frac{1}{p_x}\,\frac{\partial}{\partial x} \ln\left|\frac{p_x}{h}\right|, \quad H = \frac{1}{p_y}\,\frac{\partial}{\partial y} \ln\left|\frac{p_y}{h}\right|, \quad I = \frac{p_x\, p_y}{h}\,\cdot \tag{8.46}$$

Besides, we have four individually invariant equations (8.27) and (8.36):

$$h = 0, \quad k = 0, \quad k_x - p h_x = 0, \quad k_y - p h_y = 0. \tag{8.47}$$

8.4 Invariant differentiation and a basis of invariants. Solution of Laplace's problem

Let us find the invariant differentiations converting any invariant of equation (8.1) into invariants of the same equation. Recall that given any group with generators

$$X_\nu = \xi^i_\nu(x,u)\frac{\partial}{\partial x^i} + \eta^\alpha_\nu(x,u)\frac{\partial}{\partial u^\alpha}\,,$$

where $x = (x^1, \ldots, x^n)$ are n independent variables, there exist n invariant differentiations of the form (see [92], Chapter 7, or [39], Section 8.3.5)

$$\mathcal{D} = f^i D_i \tag{8.48}$$

where the coefficients $f^i(x, u, u_{(1)}, u_{(2)}, \ldots)$ are defined by the equations

$$X_\nu(f^i) = f^j D_j(\xi^i_\nu), \quad i = 1, \ldots, n. \tag{8.49}$$

In our case, the generators X_ν are replaced by the operators (8.22), (8.25). Let us write the invariant differential operator (8.48) in the form

$$\mathcal{D} = f\,D_x + g\,D_y. \tag{8.50}$$

The equations (8.49) for the coefficients become:

$$\begin{aligned} &X(f) = f\,D_x(\xi(x)) + g\,D_y(\xi(x)) \equiv -\xi'(x)f, \quad X(g) = 0;\\ &Y(g) = f\,D_x(\eta(y)) + g\,D_y(\eta(y)) \equiv -\eta'(y)g, \quad Y(f) = 0. \end{aligned} \tag{8.51}$$

Here f, g are unknown functions of $x, y, h, k, h_x, h_y, k_x, k_y, h_{xx}, \ldots$. The operators X and Y are prolonged to all derivatives of h, k involved here.

Let us begin with $f = f(x, y, h, k)$ and $g = g(x, y, h, k)$. Then equations (8.51) give the following equations for f :

$$\xi\frac{\partial f}{\partial x} - \xi'\Big[h\frac{\partial f}{\partial h} + k\frac{\partial f}{\partial k}\Big] = \xi'(x)f, \quad \eta\frac{\partial f}{\partial y} - \eta'\Big[h\frac{\partial f}{\partial h} + k\frac{\partial f}{\partial k}\Big] = 0.$$

Invoking that ξ, ξ', η and η' are arbitrary functions (cf. the previous section), we obtain the following four equations:

$$\frac{\partial f}{\partial x} = 0, \quad h\frac{\partial f}{\partial h} + k\frac{\partial f}{\partial k} = -f, \quad \frac{\partial f}{\partial y} = 0, \quad h\frac{\partial f}{\partial h} + k\frac{\partial f}{\partial k} = 0,$$

whence $f = 0$. Likewise, the equations (8.51) written for $g = g(x, y, h, k)$ yield $g = 0$. Thus, there are no invariant differentiations (8.50) with $f = f(x, y, h, k)$ and $g = g(x, y, h, k)$.

Therefore, we continue our search by letting

$$f = f(x, y, h, k, h_x, h_y, k_x, k_y), \quad g = g(x, y, h, k, h_x, h_y, k_x, k_y).$$

The first-order prolongations of the generators X and Y furnish the operators (cf. (8.23) and (8.26))

$$\begin{aligned} X_\xi &= \frac{\partial}{\partial x}, \qquad X_{\xi''} = h\frac{\partial}{\partial h_x} + k\frac{\partial}{\partial k_x}, \\ X_{\xi'} &= h\frac{\partial}{\partial h} + k\frac{\partial}{\partial k} + 2h_x\frac{\partial}{\partial h_x} + h_y\frac{\partial}{\partial h_y} + 2k_x\frac{\partial}{\partial k_x} + k_y\frac{\partial}{\partial k_y} \end{aligned} \tag{8.52}$$

and

$$\begin{aligned} Y_\eta &= \frac{\partial}{\partial y}, \qquad Y_{\eta''} = h\frac{\partial}{\partial h_y} + k\frac{\partial}{\partial k_y}, \\ Y_{\eta'} &= h\frac{\partial}{\partial h} + k\frac{\partial}{\partial k} + h_x\frac{\partial}{\partial h_x} + 2h_y\frac{\partial}{\partial h_y} + k_x\frac{\partial}{\partial k_x} + 2k_y\frac{\partial}{\partial k_y}, \end{aligned} \tag{8.53}$$

respectively. The operators X_ξ and X_η yield that the functions f, g do not involve the variables x and y. Furthermore, equations (8.51) split into the equations

$$X_{\xi'}(f) = -f, \quad X_{\xi''}(f) = 0, \quad Y_{\eta'}(f) = 0, \quad Y_{\eta''}(f) = 0 \tag{8.54}$$

and

$$X_{\xi'}(g) = 0, \quad X_{\xi''}(g) = 0, \quad Y_{\eta'}(g) = -g, \quad Y_{\eta''}(g) = 0 \tag{8.55}$$

for functions $f(h, k, h_x, h_y, k_x, k_y)$ and $g(h, k, h_x, h_y, k_x, k_y)$, respectively. The equations $X_{\xi''}(f) = 0, Y_{\eta''}(f) = 0$ and $X_{\xi''}(g) = 0, Y_{\eta''}(g) = 0$ yield that f and g depend on the following four variables (cf. the previous section):

$$h, \quad k, \quad \lambda = k_x - ph_x = hp_x, \quad \mu = k_y - ph_y = hp_y.$$

We rewrite the operators $X_{\xi'}$ and $Y_{\eta'}$ in the variables h, λ, μ and $p = k/h$:

$$X_{\xi'} = h\frac{\partial}{\partial h} + 2\lambda\frac{\partial}{\partial \lambda} + \mu\frac{\partial}{\partial \mu}, \quad Y_{\eta'} = h\frac{\partial}{\partial h} + \lambda\frac{\partial}{\partial \lambda} + 2\mu\frac{\partial}{\partial \mu}, \tag{8.56}$$

integrate the equations

$$X_{\xi'}(f) = -f, \quad Y_{\eta'}(f) = 0$$

and

$$X_{\xi'}(g) = 0, \quad Y_{\eta'}(g) = -g$$

for the functions $f(h,p,\lambda,\mu)$ and $g(h,p,\lambda,\mu)$, respectively, and obtain:

$$f = \frac{h}{\lambda}\,F(p,I), \quad g = \frac{h}{\mu}\,G(p,I), \tag{8.57}$$

where p and I are the invariants (8.28) and (8.44), respectively:

$$p = \frac{k}{h}\,, \quad I = \frac{\lambda\,\mu}{h^3} = \frac{p_x\,p_y}{h}\,\cdot$$

Substituting the expressions (8.57) in (8.50), one obtains the invariant differentiation

$$\mathcal{D} = F(p,I)\frac{1}{p_x}\,D_x + G(p,I)\frac{1}{p_y}\,D_y \tag{8.58}$$

with arbitrary functions $F(p,I)$ and $G(p,I)$.

Remark 5.7. The most general invariant differentiation has the form (8.58) where $F(p,I)$ and $G(p,I)$ are replaced by arbitrary functions of higher-order invariants, e.g. by $F(p,I,q_1,q_2,N,H)$ and $G(p,I,q_1,q_2,N,H)$, provided that the corresponding invariants are known. It suffices, however, to let that F and G are any constants.

Letting in (8.58) $F = 1,\ G = 0$ and then $F = 0,\ G = 1$, one obtains the following simplest invariant differentiations in directions x and y, respectively:

$$\mathcal{D}_1 = \frac{1}{p_x}\,D_x, \quad \mathcal{D}_2 = \frac{1}{p_y}\,D_y. \tag{8.59}$$

Now, one can construct higher-order invariants by means of the invariant differentiations (8.59) and prove the following statement.

Theorem 5.14. A basis of invariants of an arbitrary order for equation (8.1) is provided by the invariants

$$p = \frac{k}{h}\,, \quad I = \frac{p_x\,p_y}{h}\,, \quad q_1 = \frac{1}{h}\,\frac{\partial^2 \ln|h|}{\partial x \partial y}\,, \quad q_2 = \frac{1}{k}\,\frac{\partial^2 \ln|k|}{\partial x \partial y} \tag{8.60}$$

or, alternatively, by the invariants

$$p = \frac{k}{h}\,,\ I = \frac{p_x\,p_y}{h}\,,\ N = \frac{1}{p_x}\,\frac{\partial}{\partial x}\ln\left|\frac{p_x}{h}\right|,\ q_1 = \frac{1}{h}\,\frac{\partial^2 \ln|h|}{\partial x \partial y}\,. \tag{8.61}$$

Proof. The reckoning shows that the operators act as follows:

$$\mathcal{D}_1(p) = 1, \quad \mathcal{D}_1(I) = \Big(N + \frac{1}{p}\Big)I + p(pq_2 - q_1),$$

$$\mathcal{D}_2(p) = 1, \quad \mathcal{D}_2(I) = \Big(H + \frac{1}{p}\Big) I + p(pq_2 - q_1).$$

Hence, the invariants (8.61) can be obtained from (8.60) by invariant differentiations, and vice versa. Consequently, a basis of all invariants of the second order (8.45)-(8.46) is provided by (8.60) or by (8.61). Furthermore, one can show, invoking equations (8.37), that the invariants differentiations $\mathcal{D}_1$ and $\mathcal{D}_2$ of the basic invariants (8.60), or (8.61) provide 6 independent invariants involving third-order partial derivatives of h and k. On the other hand, consideration of third-order invariants involves 8 third-order derivatives of of h and k. However, the invariance condition brings two additional equations due to the fourth-order derivatives $\xi^{(iv)}(x)$ and $\eta^{(iv)}(y)$, so that we will have precisely 6 additional invariants, just as given by invariant differentiations. The same reasoning for higher-order derivatives completes the proof.

8.5 Alternative representation of invariants

It can be useful to write the invariants of the hyperbolic equations in the coordinates

$$z = x + y, \quad t = x - y. \tag{8.62}$$

We have:

$$x = \frac{z+t}{2}, \quad y = \frac{z-t}{2},$$

$$u_x = u_z + u_t, \quad u_y = u_z - u_t, \quad u_{xy} = u_{zz} - u_{tt}.$$

Then Eq. (8.1) is written in the alternative standard form:

$$u_{zz} - u_{tt} + \tilde{a}(z,t)u_z + \tilde{b}(z,t)u_t + \tilde{c}(z,t)u = 0, \tag{8.63}$$

where

$$\begin{aligned} \tilde{a}(z,t) &= a(x,y) + b(x,y), \\ \tilde{b}(z,t) &= a(x,y) - b(x,y), \\ \tilde{c}(z,t) &= c(x,y). \end{aligned} \tag{8.64}$$

The equivalence algebra $L_{\mathcal{E}}$ for Eq. (8.63) can be obtained without solving again the determining equations. Rather we obtain it by rewriting the generator (8.3) in the alternative coordinates (8.62). Let us consider, for the sake of brevity, the (x, y, u) part of the operator (8.3) and write it as follows:

$$Y = \xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y} + \sigma\, u\frac{\partial}{\partial u}, \quad \xi_y = 0, \ \eta_x = 0.$$

Rewriting it in the new variables (8.62), we have:

$$Y = \widetilde{\xi}\frac{\partial}{\partial z} + \widetilde{\eta}\frac{\partial}{\partial t} + \widetilde{\sigma}\, u\frac{\partial}{\partial u}\,,$$

where

$$\widetilde{\xi} = \xi + \eta, \quad \widetilde{\eta} = \xi - \eta, \quad \widetilde{\sigma} = \sigma.$$

Now we rewrite the conditions $\xi_y = 0,\ \eta_x = 0$ in terms of $\overline{\xi}(z,t), \overline{\eta}(z,t)$. We have

$$\xi = \frac{1}{2}(\widetilde{\xi} + \widetilde{\eta}), \quad \eta = \frac{1}{2}(\widetilde{\xi} - \widetilde{\eta})$$

and

$$\xi_y = \frac{1}{2}(\widetilde{\xi}_z + \widetilde{\eta}_z - \widetilde{\xi}_t - \widetilde{\eta}_t), \quad \eta_x = \frac{1}{2}(\widetilde{\xi}_z - \widetilde{\eta}_z + \widetilde{\xi}_t - \widetilde{\eta}_t).$$

Hence, the conditions $\xi_y = 0,\ \eta_x = 0$ become:

$$\widetilde{\xi}_z + \widetilde{\eta}_z - \widetilde{\xi}_t - \widetilde{\eta}_t = 0, \quad \widetilde{\xi}_z - \widetilde{\eta}_z + \widetilde{\xi}_t - \widetilde{\eta}_t = 0,$$

whence

$$\widetilde{\xi}_z = \widetilde{\eta}_t, \quad \widetilde{\eta}_z = \widetilde{\xi}_t.$$

This proves the following theorem.

Theorem 5.15. The equivalence algebra $L_{\mathcal{E}}$ for the family of equations (8.63) comprises the operators (properly extended to the coefficients $\tilde{a}, \tilde{b}, \tilde{c}$)

$$Y = \widetilde{\xi}(z,t)\frac{\partial}{\partial z} + \widetilde{\eta}(z,t)\frac{\partial}{\partial t} + \widetilde{\sigma}(z,t)\, u\frac{\partial}{\partial u}\,, \tag{8.65}$$

where $\widetilde{\beta}(z,t)$ is an arbitrary function, whereas $\widetilde{\xi}(z,t), \widetilde{\eta}(z,t)$ solve the equations

$$\widetilde{\xi}_z = \widetilde{\eta}_t, \quad \widetilde{\eta}_z = \widetilde{\xi}_t. \tag{8.66}$$

The operator (8.65) generates the continuous infinite group $\mathcal{E}_c$ of equivalence transformations composed of the linear transformation of the dependent variable:

$$u = \psi(z,t)\, v, \quad \psi(x,y) \neq 0,$$

and the following change of the independent variables:

$$\overline{z} = f\Big(\frac{z+t}{2}\Big) + g\Big(\frac{z-t}{2}\Big), \quad \overline{t} = f\Big(\frac{z+t}{2}\Big) - g\Big(\frac{z-t}{2}\Big).$$

The invariants of Eq. (8.63) are readily obtained by applying the transformation (8.62) to the invariants invariants of the previous section. Namely, we have from (8.64):

$$a = \frac{\tilde{a} + \tilde{b}}{2}, \quad b = \frac{\tilde{a} - \tilde{b}}{2}, \quad c = \tilde{c},$$

and hence, invoking (8.62):

$$a_x = \frac{1}{2}\Big(\tilde{a}_z + \tilde{a}_t + \tilde{b}_z + \tilde{b}_t\Big), \quad b_y = \frac{1}{2}\Big(\tilde{a}_z - \tilde{a}_t - \tilde{b}_z + \tilde{b}_t\Big).$$

The Laplace invariants (8.17) become

$$\begin{aligned} h &= \frac{1}{2}\Big(\tilde{a}_z + \tilde{a}_t + \tilde{b}_z + \tilde{b}_t\Big) + \frac{1}{4}\Big(\tilde{a}^2 - \tilde{b}^2\Big) - \tilde{c}, \\ k &= \frac{1}{2}\Big(\tilde{a}_z - \tilde{a}_t - \tilde{b}_z + \tilde{b}_t\Big) + \frac{1}{4}\Big(\tilde{a}^2 - \tilde{b}^2\Big) - \tilde{c}. \end{aligned}$$

It is convenient to take their linear combinations $h + k, h - k$ and obtain the following semi-invariants for Eq. (8.62):

$$\tilde{h} = \tilde{a}_z + \tilde{b}_t + \frac{1}{2}\Big(\tilde{a}^2 - \tilde{b}^2\Big) - 2\tilde{c}, \quad \tilde{k} = \tilde{a}_t + \tilde{b}_z. \tag{8.67}$$

We have

$$h = \frac{1}{2}(\tilde{h} + \tilde{k}), \quad k = \frac{1}{2}(\tilde{h} - \tilde{k}), \tag{8.68}$$

and Ovsyannikov's invariant p (see (8.60)) is written in the form

$$p = \frac{k}{h} = \frac{\tilde{h} - \tilde{k}}{\tilde{h} + \tilde{k}} \equiv \frac{1 - (\tilde{k}/\tilde{h})}{1 + (\tilde{k}/\tilde{h})} \cdot \tag{8.69}$$

It follows that

$$\frac{\tilde{k}}{\tilde{h}} = \frac{1 - p}{1 + p} \cdot$$

Thus, we get the following Ovsyannikov's invariant for Eq. (8.63):

$$\tilde{p} = \frac{\tilde{k}}{\tilde{h}} \cdot \tag{8.70}$$

Likewise, we can rewrite all invariants (8.45)-(8.46) in the alternative coordinates and obtain the invariants for Eq. (8.63). For example, the invariant I from (8.46), is rewritten using (8.68) as follows:

$$I = \frac{p_x\, p_y}{h} = \frac{2}{\tilde{h} + \tilde{k}}\Big(p_z + p_t\Big)\Big(p_z - p_t\Big) = \frac{2}{\tilde{h} + \tilde{k}}\Big(p_z^2 - p_t^2\Big). \tag{8.71}$$

Now we get, use the expression (8.68) for p :

$$p_z = 2\,\frac{\tilde{k}\tilde{h}_z - \tilde{h}\tilde{k}_z}{(\tilde{h}+\tilde{k})^2}\,, \quad p_t = 2\,\frac{\tilde{k}\tilde{h}_t - \tilde{h}\tilde{k}_t}{(\tilde{h}+\tilde{k})^2}\,, \tag{8.72}$$

and substitute in (8.71) to obtain the following expression for the invariant (8.71):

$$I = \frac{8}{(\tilde{h}+\tilde{k})^5}\Big[(\tilde{k}\tilde{h}_z - \tilde{h}\tilde{k}_z)^2 - (\tilde{k}\tilde{h}_t - \tilde{h}\tilde{k}_t)^2\Big].$$

Since $\tilde{h}+\tilde{k} = (\tilde{p}+1)\tilde{k}$ and $\tilde{p}$ is an invariant (see (8.70)), we can replace $\tilde{h}+\tilde{k}$ by $\tilde{k}$ and, ignoring the constant coefficient, obtain the following invariant for Eq. (8.63):

$$\widetilde{I} = \frac{1}{\tilde{k}^5}\Big[(\tilde{k}\tilde{h}_z - \tilde{h}\tilde{k}_z)^2 - (\tilde{k}\tilde{h}_t - \tilde{h}\tilde{k}_t)^2\Big]. \tag{8.73}$$

Furthermore, we can readily obtain from (8.59) the corresponding invariant differentiations for Eq. (8.63). We have:

$$\mathcal{D}_1 = \frac{1}{p_x}\,D_x = \frac{1}{p_x}\,\Big(D_z + D_t\Big), \quad \mathcal{D}_2 = \frac{1}{p_y}\,D_y = \frac{1}{p_y}\,\Big(D_z - D_t\Big).$$

Substituting here the following expressions for p_x and p_y (see (8.72)):

$$p_x = p_z + p_t = \frac{2}{(\tilde{h}+\tilde{k})^2}\Big[\tilde{k}(\tilde{h}_z + \tilde{h}_t) - \tilde{h}(\tilde{k}_z + \tilde{k}_t)\Big]$$

$$p_y = p_z - p_t = \frac{2}{(\tilde{h}+\tilde{k})^2}\Big[\tilde{k}(\tilde{h}_z - \tilde{h}_t) - \tilde{h}(\tilde{k}_z - \tilde{k}_t)\Big],$$

and ignoring the coefficient 2, we obtain:

$$\mathcal{D}_1 = \frac{(\tilde{h}+\tilde{k})^2}{\tilde{k}(\tilde{h}_z + \tilde{h}_t) - \tilde{h}(\tilde{k}_z + \tilde{k}_t)}\,\Big(D_z + D_t\Big),$$

$$\mathcal{D}_2 = \frac{(\tilde{h}+\tilde{k})^2}{\tilde{k}(\tilde{h}_z - \tilde{h}_t) - \tilde{h}(\tilde{k}_z - \tilde{k}_t)}\,\Big(D_z - D_t\Big).$$

Since $\tilde{h} = \tilde{p}\,\tilde{k}$ (see (8.70)), and hence $\tilde{h}+\tilde{k} = (1+\tilde{p})\tilde{k}$, the invariant differentiations for Eq. (8.63) can be written as follows:

$$\widetilde{\mathcal{D}}_1 = \frac{\tilde{k}}{\tilde{h}_z + \tilde{h}_t - \tilde{p}(\tilde{k}_z + \tilde{k}_t)}\,\Big(D_z + D_t\Big),$$

$$\widetilde{\mathcal{D}}_2 = \frac{\tilde{k}}{\tilde{h}_z - \tilde{h}_t - \tilde{p}(\tilde{k}_z - \tilde{k}_t)}\,\Big(D_z - D_t\Big). \tag{8.74}$$

9 Invariants of linear elliptic equations

E. Cotton [14] extended the Laplace invariants to the elliptic equations

$$u_{\alpha\alpha} + u_{\beta\beta} + A(\alpha,\beta)u_\alpha + B(\alpha,\beta)u_\beta + C(\alpha,\beta)u = 0 \tag{9.1}$$

and obtained the following semi-invariants:

$$H = A_\alpha + B_\beta + \frac{1}{2}\Big(A^2 + B^2\Big) - 2C, \quad K = A_\beta - B_\alpha. \tag{9.2}$$

Cotton's invariants can be derived by considering the linear transformation (8.9) and proceeding as in Section 8.2. This way was illustrated in [41] where I also mentioned that Cotton's invariants (9.2) can be obtained from the Laplace invariants (8.17) merely by a complex change of the independent variables connecting the hyperbolic and elliptic equations. Namely, the change of variables

$$\alpha = x + y, \quad \beta = i(y - x) \tag{9.3}$$

maps the hyperbolic equation (8.1) into the elliptic equation (9.1). We will rather use the alternative representation of invariants given in Section 8.5. Then the variables (9.3) and (8.62) are related by

$$\alpha = z, \quad \beta = -it. \tag{9.4}$$

It is manifest that

$$D_z = D_\alpha, \quad D_t = -i\,D_\beta \tag{9.5}$$

and the hyperbolic equation (8.63) becomes the elliptic equation (9.1), where

$$A = \tilde{a}, \quad B = -i\,\tilde{b}, \quad C = \tilde{c}. \tag{9.6}$$

Theorem 5.16. The equivalence algebra $L_\mathcal{E}$ for the family of the elliptic equations (9.1) comprises the operators (properly extended to the coefficients A, B, C)

$$Y = \xi^1(\alpha,\beta)\frac{\partial}{\partial\alpha} + \xi^2(\alpha,\beta)\frac{\partial}{\partial\beta} + \nu(\alpha,\beta)\,u\frac{\partial}{\partial u}\,, \tag{9.7}$$

where $\nu(\alpha,\beta)$ is an arbitrary function, whereas $\xi^1(\alpha,\beta), \xi^2(\alpha,\beta)$ solve the Cauchy-Riemann equations

$$\xi^1_\alpha = \xi^2_\beta, \quad \xi^2_\alpha = -\xi^1_\beta. \tag{9.8}$$

Proof. We proceed as in Theorem 5.15. Namely, we rewrite the operator (8.65) in the new variables (9.4) to obtain

$$Y = \xi^1(\alpha,\beta)\frac{\partial}{\partial\alpha} + \xi^2(\alpha,\beta)\frac{\partial}{\partial\beta} + \nu(\alpha,\beta)\, u\frac{\partial}{\partial u}\,,$$

where

$$\xi^1 = \widetilde{\xi}, \quad \xi^2 = -i\,\widetilde{\eta}, \quad \nu = \widetilde{\sigma}.$$

We have

$$\widetilde{\xi}_z = \xi^1_\alpha, \quad \widetilde{\xi}_t = -i\,\xi^1_\beta, \quad \widetilde{\eta}_z = i\,\xi^2_\alpha, \quad \widetilde{\eta}_t = i\,\xi^2_\beta.$$

It follows that the equations (8.66) become the Cauchy-Riemann system (9.8), thus proving the theorem.

Using (9.5) and (9.6), we readily obtain the Cotton invariants (9.2) from the semi-invariants (8.67). Namely, they are related as follows:

$$\tilde{h} = H, \quad \tilde{k} = -i\,K. \tag{9.9}$$

Likewise we transform the invariants $\tilde{p}$ (8.70) and $\widetilde{I}$ (8.73) for the hyperbolic equation (8.63) into the following invariants for the elliptic equation (9.1):

$$P = \frac{K}{H} \tag{9.10}$$

and

$$J = \frac{1}{K^5}\Big[(KH_z - HK_z)^2 + (KH_t - H.K_t)^2\Big]. \tag{9.11}$$

We have the relations

$$\tilde{p} = -i\,P, \quad \widetilde{I} = -J. \tag{9.12}$$

It is not difficult to transform in a similar way all basic invariants (8.60) and (8.61) as well as the individually invariant equations (8.47) and obtain the invariants and the individually invariant equations, respectively, for the elliptic equation (9.1).

Furthermore, one can readily obtain from the invariant differentiations (8.74) the corresponding invariant differentiations for Eq. (9.1). Indeed, using Eqs. (9.5), (9.9) and (9.12), we rewrite the operators (8.74) as follows:

$$\widetilde{\mathcal{D}}_1 = \frac{K}{(H_\beta + PK_\beta) + i(H_\alpha + PK_\alpha)}\Big(D_\alpha - iD_\beta\Big),$$

$$\widetilde{\mathcal{D}}_2 = \frac{-K}{(H_\beta + PK_\beta) - i(H_\alpha + PK_\alpha)}\Big(D_\alpha + iD_\beta\Big),$$

or

$$\widetilde{\mathcal{D}}_1 = S\left[(Q_\beta D_\alpha - Q_\alpha D_\beta) - i(Q_\alpha D_\alpha + Q_\beta D_\beta)\right],$$

$$\widetilde{\mathcal{D}}_2 = -S\left[(Q_\beta D_\alpha - Q_\alpha D_\beta) + i(Q_\alpha D_\alpha + Q_\beta D_\beta)\right],$$

where

$$Q_\alpha = H_\alpha + PK_\alpha, \quad Q_\beta = H_\beta + PK_\beta, \quad S = \frac{K}{Q_\alpha^2 + Q_\beta^2} \cdot$$

Singling out the real and imaginary parts obtained by taking the linear combinations

$$\widehat{\mathcal{D}}_1 = \frac{1}{2}\Big(\widetilde{\mathcal{D}}_1 - \widetilde{\mathcal{D}}_2\Big), \quad \widehat{\mathcal{D}}_2 = \frac{i}{2}\Big(\widetilde{\mathcal{D}}_1 + \widetilde{\mathcal{D}}_2\Big),$$

we arrive at the following invariant differentiations for Equation (9.1):

$$\widehat{\mathcal{D}}_1 = \frac{K}{(H_\beta + PK_\beta)^2 + (H_\alpha + PK_\alpha)^2}\Big[(H_\beta + PK_\beta)D_\alpha - (H_\alpha + PK_\alpha)D_\beta\Big],$$

$$\widehat{\mathcal{D}}_2 = \frac{K}{(H_\beta + PK_\beta)^2 + (H_\alpha + PK_\alpha)^2}\Big[(H_\alpha + PK_\alpha)D_\alpha + (H_\beta + PK_\beta)D_\beta\Big].$$

One can rewrite in a similar way the basic invariants (8.60) and (8.61) as well as the individually invariant equations (8.47). Subjecting these basic invariants to the invariant differentiations $\widehat{\mathcal{D}}_1$ and $\widehat{\mathcal{D}}_2$, one obtains all invariants of the elliptic equations.

10 Semi-invariants of parabolic equations

Consider the parabolic equations written in the canonical form:

$$u_t + a(t,x)u_{xx} + b(t,x)u_x + c(t,x)u = 0. \tag{10.1}$$

The group of equivalence transformations of the equations (10.1) is an infinite group composed of linear transformation of the dependent variable:

$$u = \sigma(x,y)v, \quad \sigma(x,y) \neq 0, \tag{10.2}$$

and invertible changes of the independent variables of the form:

$$\tau = \phi(t), \quad y = \psi(t,x), \tag{10.3}$$

where $\phi(t), \psi(t,x), \sigma(t,x)$ are arbitrary functions. Let us verify that (10.2)–(10.3) are equivalence transformations. The total differentiations

$$D_t = \frac{\partial}{\partial t} + u_t\frac{\partial}{\partial u} + u_{tt}\frac{\partial}{\partial u_t} + u_{tx}\frac{\partial}{\partial u_x} + \cdots,$$

$$D_x = \frac{\partial}{\partial x} + u_x\frac{\partial}{\partial u} + u_{tx}\frac{\partial}{\partial u_t} + u_{xx}\frac{\partial}{\partial u_x} + \cdots$$

are mapped by (10.3) to the operators D_τ, D_y defined by the equations

$$D_t = D_t(\phi)D_\tau + D_t(\psi)D_y, \quad D_x = D_x(\psi)D_y,$$

or

$$D_t = \phi'(t)D_\tau + \psi_t D_y, \quad D_x = \psi_x D_y. \tag{10.4}$$

Application of (10.4) to (10.2) yields:

$$u_t = \sigma\phi' v_\tau + \sigma\psi_t v_y + \sigma_t v, \quad u_x = \sigma\psi_x v_y + \sigma_x v,$$

$$u_{xx} = \sigma\psi_x^2 v_{yy} + (\sigma\psi_{xx} + 2\sigma_x\psi_x)v_y + \sigma_{xx}v.$$

Hence, the equation (10.1) is transferred by the transformations (10.2)–(10.3) to an equation having again the form (10.1):

$$\begin{aligned}\phi' v_\tau + a\psi_x^2 v_{yy} + \Big[\Big(\psi_{xx} + 2\frac{\sigma_x}{\sigma}\psi_x\Big)a + b\psi_x + \psi_t\Big]v_y&\\ + \Big[\frac{\sigma_t}{\sigma} + \frac{\sigma_{xx}}{\sigma}a + \frac{\sigma_x}{\sigma}b + c\Big]v = 0.&\end{aligned} \tag{10.5}$$

Let us find *semi-invariants* of the equations (10.1), i.e. its invariants under the transformation (10.2). Substituting the infinitesimal transformation

$$u \approx [1 + \varepsilon\eta(x,y)]v$$

into the equation (10.1), or using (10.5) with

$$\phi(t) = t, \quad \psi(t,x) = x,$$

one obtains:

$$v_t + av_{xx} + (b + 2\varepsilon a\eta_x)v_x + [c + \varepsilon(\eta_t + a\eta_{xx} + b\eta_x)]v = 0.$$

Thus, the coefficients of the equation (10.1) undergo the infinitesimal transformations

$$\overline{a} = a, \quad \overline{b} \approx b + 2\varepsilon a\eta_x, \quad \overline{c} \approx c + \varepsilon(\eta_t + a\eta_{xx} + b\eta_x), \tag{10.6}$$

that provide the generator

$$X = 2a\eta_x \frac{\partial}{\partial b} + \Big(\eta_t + a\eta_{xx} + b\eta_x\Big)\frac{\partial}{\partial c}\cdot \tag{10.7}$$

The infinitesimal test $XJ = 0$ for the invariants $J(a, b, c)$ is written

$$2a\eta_x \frac{\partial J}{\partial b} + \Big(\eta_t + a\eta_{xx} + b\eta_x\Big)\frac{\partial J}{\partial c} = 0,$$

whence $\partial J/\partial c = 0,\ \partial J/\partial b = 0$. Hence, the only independent solution is

$$J = a. \tag{10.8}$$

Therefore, we consider the semi-invariants involving the first-order derivatives (first-order differential invariants for the operator (10.7)), i.e. those of the form

$$J(a, a_t, a_x; b, b_t, b_x; c, c_t, c_x).$$

The once-extended generator (10.7) is:

$$X = 2a\eta_x \frac{\partial}{\partial b} + \Big(\eta_t + a\eta_{xx} + b\eta_x\Big)\frac{\partial}{\partial c} + 2\Big(a\eta_{tx} + a_t\eta_x\Big)\frac{\partial}{\partial b_t} + 2\Big(a\eta_{xx} + a_x\eta_x\Big)\frac{\partial}{\partial b_x}$$

$$+\Big(\eta_{tt} + a\eta_{txx} + a_t\eta_{xx} + b\eta_{tx} + b_t\eta_x\Big)\frac{\partial}{\partial c_t} + \Big(\eta_{tx} + a\eta_{xxx} + a_x\eta_{xx} + b\eta_{xx} + b_x\eta_x\Big)\frac{\partial}{\partial c_x}\cdot$$

The equation

$$XJ = 0,$$

upon equating to zero the terms with $\eta_{txx},\ \eta_{xxx},\ \eta_{tx},\ \eta_t,\ \eta_{xx}$, and finally with η_x yields

$$\frac{\partial J}{\partial c_t} = 0, \quad \frac{\partial J}{\partial c_x} = 0, \quad \frac{\partial J}{\partial b_t} = 0, \quad \frac{\partial J}{\partial c} = 0, \quad \frac{\partial J}{\partial b_x} = 0, \quad \frac{\partial J}{\partial b} = 0.$$

It follows that

$$J = J(a, a_t, a_x). \tag{10.9}$$

Thus, there are no first-order differential invariants other than the trivial ones, i.e. $J = J(a, a_t, a_x)$. Therefore, let us look for the semi-invariants of the second order (second-order differential invariants) , i.e. those of the form

$$J(a, a_t, a_x, a_{tt}, a_{tx}, a_{xx}; b, b_t, b_x, b_{tt}, b_{tx}, b_{xx}; c, c_t, c_x; c_{tt}, c_{tx}, c_{xx}).$$

We take the twice-extended generator (10.7) and proceed as above. Then we first arrive at the equations

$$\begin{gathered}\frac{\partial J}{\partial c_{tt}}=0, \quad \frac{\partial J}{\partial c_{tx}}=0, \quad \frac{\partial J}{\partial c_{xx}}=0, \\ \frac{\partial J}{\partial b_{tt}}=0, \quad \frac{\partial J}{\partial b_{tx}}=0, \quad \frac{\partial J}{\partial c_{t}}=0, \quad \frac{\partial J}{\partial c}=0.\end{gathered} \tag{10.10}$$

Eqs. (10.10) yield that

$$J=J(a, a_t, a_x, a_{tt}, a_{tx}, a_{xx}; b, b_t, b_x, b_{xx}; c_x),$$

and the equation $XJ=0$ reduces to the following system of four equations:

$$\frac{\partial J}{\partial c_x}+2a\frac{\partial J}{\partial b_t}=0, \quad a\frac{\partial J}{\partial b_t}-\frac{\partial J}{\partial b_{xx}}=0, \quad a\frac{\partial J}{\partial b_x}+(a_x-b)\frac{\partial J}{\partial b_{xx}}=0,$$

$$a\frac{\partial J}{\partial b}+a_t\frac{\partial J}{\partial b_t}+a_x\frac{\partial J}{\partial b_x}+(a_{xx}-b_x)\frac{\partial J}{\partial b_{xx}}=0. \tag{10.11}$$

Solving Eqs. (10.11) we arrive at the following result obtained in [41].

Theorem 5.17. The semi-invariants of the second order for the family of parabolic equations (10.1) have the form

$$J=\Phi(K, a, a_t, a_x, a_{tt}, a_{tx}, a_{xx}), \tag{10.12}$$

where Φ is an arbitrary function and

$$K=\frac{1}{2}\,b^2 a_x+\Big(a_t+aa_{xx}-a_x^2\Big)b+(aa_x-ab)b_x-ab_t-a^2 b_{xx}+2a^2 c_x. \tag{10.13}$$

Hence, the quantity K given by (10.13) is the main semi-invariant and furnishes, together with a, a basis of the second-order semi-invariants.

Remark 5.8. In addition to the semi-invariants (10.12), there is an invariant equation with respect to the equivalence transformations (10.2)–(10.3). This singular invariant equation is derived in [69] and involves the derivatives of the coefficient a up to fifth-order and the derivatives of K with respect to x up to second order.

Proceeding as in the case of hyperbolic equations, one can prove the following theorem stating that the semi-invariants involving higher-order derivatives of a, b, c are obtained merely by differentiating a and K.

Theorem 5.18. The general semi-invariant for the parabolic equations (10.1) has the following form:

$$J=J(t, x, a, K, a_t, a_x, K_t, K_x, a_{tt}, a_{tx}, a_{xx}, K_{tt}, K_{tx}, K_{xx}, \ldots). \tag{10.14}$$

11 Invariants of nonlinear wave equations

11.1 Equations $v_{tt} = f(x, v_x)v_{xx} + g(x, v_x)$

In [68], the second-order invariants are obtained for Eqs. (4.1),

$$v_{tt} = f(x, v_x)v_{xx} + g(x, v_x).$$

The following invariants with respect to the infinite equivalence algebra (4.11) provides a basis of invariants:

$$\lambda = \frac{f f_{22}}{(f_2)^2}\,,$$

$$\mu = \frac{f f_{22}(2\,g_2 - f_1) - f f_2 f_{12} - 3\,(f_2)^2\,(g_2 - f_1)}{f_2\,[f_2\,(g_2 - f_1) + f\,(f_{12} - g_{22})]}\,,$$

$$\nu = f\,\frac{f_1(f_1 f_{22} + 2 f_2 g_{22}) + 4\,g_2\,[f_{22}(g_2 - f_1) - f_2 g_{22}]}{[f_2\,(g_2 - f_1) + f\,(f_{12} - g_{22})]^2}$$
$$-\,(f_2)^2\,\frac{2\,[(f_1)^2 + (g_2)^2] + f(f_{11} - 2g_{12}) + f_2 g_1 - 5\,f_1 g_2}{[f_2\,(g_2 - f_1) + f\,(f_{12} - g_{22})]^2}\,.$$

Here the subscripts denote the respective differentiations:

$$f_1 = \frac{\partial f}{\partial x}\,, \quad f_2 = \frac{\partial f}{\partial v_x}\,, \ldots .$$

Furthermore, the following four individually invariant equations are singled out:

$$f_2 \equiv \frac{\partial f}{\partial v_x} = 0,$$

$$f_2\,(g_2 - f_1) + f\,(f_{12} - g_{22}) = 0,$$

$$f f_{22}(2\,g_2 - f_1) - f f_2 f_{12} - 3\,(f_2)^2\,(g_2 - f_1) = 0,$$

$$f\,\{f_1(f_1 f_{22} + 2 f_2 g_{22}) + 4\,g_2\,[f_{22}(g_2 - f_1) - f_2 g_{22}]\}$$
$$-\,f_2^2\{2[(f_1)^2 + (g_2)^2] + f(f_{11} - 2g_{12}) + f_2 g_1 - 5\,f_1 g_2\} = 0.$$

11.2 Equations $u_{tt} - u_{xx} = f(u, u_t, u_x)$

It shown in [105] that Eqs. (4.15),

$$u_{tt} - u_{xx} = f(u, u_t, u_x)$$

have the following first-order invariant:

$$\frac{2\,f-(u_t-u_x)(f_{u_t}-f_{u_x})}{2\,f-(u_t+u_x)(f_{u_t}+f_{u_x})},$$

and two individually invariant equations:

$$(u_t-u_x)(f_{u_t}-f_{u_x})-2f=0$$

and

$$(u_t+u_x)(f_{u_t}+f_{u_x})-2f=0.$$

12 Invariants ofevolution equations

12.1 Generalised Burgers equations

It is shown in [63], that the generalised Burgers equation (5.1):

$$u_t+uu_x+f(t)u_{xx}=0,$$

has for its minimal-order invariant a third-order invariant, namely the *Schwarzian*

$$J=\frac{f^2}{f'^3}\left[f'''-\frac{3}{2}\,\frac{f''^2}{f'}\right].$$

Moreover, it has the following invariant differentiation:

$$\mathcal{D}_t=\frac{f}{f'}D_t.$$

It is restricted to differentiation in t since f depends only on t.

The following generalization of the Burgers equation is also considered in [63]:

$$u_t+uu_x+g(t,x)u_{xx}=0. \tag{12.1}$$

It is shown that the generators (5.5) span the equivalence algebra for Eq. (12.1) as well and have the following invariants:

$$J_1=\frac{f_x^2}{f\,f_{xx}},\quad J_2=\frac{f^2}{f_{xx}^3}(2f_tf_xf_{tx}-f_t^2f_{xx}-f_x^2f_{tt}).$$

12.2 Equation $u_t = u_{xx} + g(x, u, u_x)$

The following three third-order invariants for this equation are found in [65]:

$$J_1 = \frac{I_1 I_3}{I_2^2}, \quad J_2 = \frac{I_1 I_4^3}{I_2^5}, \quad J_3 = \frac{I_1^2 I_5^3}{I_2^4},$$

where

$$\begin{aligned}
I_1 &= g_{u_x u_x u_x}, \\
I_2 &= g g_{u_x u_x u_x} + g_{u u_x} - g_{u u_x u_x} u_x - g_{x u_x u_x}, \\
I_3 &= g^2 g_{u_x u_x u_x} + g g_{u u_x} - 2 g g_{u u_x u_x} u_x - 2 g g_{x u_x u_x} \\
&- g_u g_{u_x u_x} u_x + g_{u_x} g_{u u_x} u_x + g_{u_x} g_{x u_x} - g_{u_x u_x} g_x \\
&- 2 g_{uu} u_x + g_{u u u_x} u_x^2 - 2 g_{xu} 2 g_{x u u_x} u_x + g_{x x u_x}, \qquad (12.2) \\
I_4 &= g g_{u_x} g_{u_x u_x u_x} - 2 g g_{u u_x u_x} - 2 g_u g_{u_x u_x} + 3 g_{u_x} g_{u u_x} \\
&- g_{u_x} g_{u u_x u_x} u_x - g_{u_x} g_{x u_x u_x} - 4 g_{uu} + 2 g_{u u u_x} u_x + 2 g_{x u u_x}, \\
I_5 &= 2 g g_{u_x u_x} - 2 u_x g_{u u_x} - 2 g_{x u_x} - g_{u_x}^2 + 4 g_u.
\end{aligned}$$

Furthermore, it is demonstrated in [65] that the equation under consideration has the following six invariant equations:

$$2 g g_{u_x u_x} - 2 u_x g_{u u_x} - 2 g_{x u_x} - g_{u_x}^2 + 4 g_u = 0$$

and

$$I_1 = 0, \quad I_2 = 0, \quad I_3 = 0, \quad I_4 = 0, \quad I_5 = 0$$

with $I_1, \ldots, I_5$ defined in (12.2).

12.3 Equation $u_t = f(x, u) u_{xx} + g(x, u, u_x)$

This equation has the following two second-order invariants ([64]):

$$J_1 = \frac{I_1}{f_u^2}, \quad J_2 = \frac{f_u^2 I_2}{I_3^2},$$

where

$$\begin{aligned}
I_1 &= f_u g_{u_x u_x} - 2 f f_{uu}, \\
I_2 &= 3 u_x^2 f_u^2 - 4 u_x^2 f f_{uu} - 8 u_x f f_{xu} - 4 f f_{xx} - 16 f g_u + 8 u_x f g_{u u_x} + 8 f g_{x u_x} \\
&\quad + 6 u_x f_x f_u + 20 g f_u - 8 u_x f_u g_{u_x} + 3 f_x^2 - 8 f_x g_{u_x} - 8 g g_{u_x u_x} + 4 g_{u_x}^2, \\
I_3 &= 4 u_x f f_{uu} + 4 f f_{xu} - 3 u_x f_u^2 - 3 f_x f_u - 2 f_u g_{u_x}.
\end{aligned}$$

Furthermore, it is shown in [64] that the equation under consideration has the following six invariant equations:

$$f_u = 0, \quad I_1 = 0, \quad I_2 = 0, \quad I_3 = 0.$$

12.4 Equation $u_t = f(x, u, u_x)u_{xx} + g(x, u, u_x)$

Here the second-order invariant is

$$J = \frac{I_1 I_3}{I_2^2}\,,$$

where

$$\begin{aligned} I_1 &= 2ff_{u_x u_x} - 3f_{u_x}^2, \\ I_2 &= \left[2ff_{u_x u_x} - 3f_{u_x}^2\right] g - 2f^2\left[u_x f_{uu_x} + f_{xu_x}\right] + 3u_x f f_{u_x} f_u \\ &+ 2f^2 f_u + 3ff_{u_x} f_x, \\ I_3 &= 8f^2 g g_{u_x u_x} - 8u_x f^3 g_{uu_x} - 8f^3 g_{xu_x} - 4fg^2 f_{u_x u_x} + 4u_x^2 f^3 f_{uu} \\ &+ 8u_x f^3 f_{xu} + 4f^3 f_{xx} - 4f^2 g_{u_x}^2 + 4f\left(2u_x f f_u - g f_{u_x} + 2ff_x\right) g_{u_x} \\ &+ 4f^2\left(u_x f_{u_x} + 4f\right) g_u + 4f^2 f_{u_x} g_x + 5g^2 f_{u_x}^2 - 6fg(u_x f_u + f_x) f_{u_x} \\ &- 3u_x^2 f^2 f_u^2 - 2f^2(3u_x f_x + 10g) f_u - 3f^2 f_x^2. \end{aligned}$$

The equations

$$I_1 = 0, \quad I_2 = 0, \quad I_3 = 0$$

are invariant under the equivalence group with the generators (5.8).

Paper 6

Euler's integration method for linear hyperbolic equations and its extension to parabolic equations

Revised paper [52]

Abstract. The first significant results towards the general integration theory for hyperbolic equations with two independent variables were obtained by Leonard Euler. He generalized d'Alembert's solution to a wide class of linear hyperbolic equations with two independent variables and introduced the quantities that were rediscovered by Laplace and known today as the Laplace invariants. In the present paper, I give an overview of Euler's method for hyperbolic equations and then extend Euler's method to parabolic equations. The new method, based on the invariant of parabolic equations, allows one to identify all linear parabolic equations reducible to the heat equation and find their general solution. The method is illustrated by the Black-Scholes equation for which the general solution and the solution of an arbitrary Cauchy problem are provided.

1 Introduction

When I studied partial differential equations at university, one of amazing facts for me was that one could integrate the wave equation

$$u_{tt} - u_{xx} = 0 \tag{1.1}$$

and obtain explicitly its general solution

$$u = f(x + t) + g(x - t),$$

while the general solution is not given in university texts for two other ubiquitous equations: the heat equation

$$v_t - v_{xx} = 0 \tag{1.2}$$

and the Laplace equation

$$u_{xx} + u_{yy} = 0. \tag{1.3}$$

Later it become clear that this was partially due to a significant difference between the three classical equations in terms of characteristics. Namely, the wave equation has two families of real characteristics and can be written in the characteristic variables $\xi = x+t,\ \eta = x-t$ in the *factorable* form:

$$u_{\xi\eta} \equiv D_\eta D_\xi u = 0. \tag{1.4}$$

Therefore it can be solved by consecutive integration of two first-order linear ordinary differential equations:

$$D_\eta v = 0, \quad D_\xi u = v.$$

The integration can also be done by writing Eq. (1.1) in the factorized form:

$$\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right) u = 0.$$

A similar factorization to first-order differential operators is impossible for the heat and Laplace operators since the heat equation has only one family of characteristics whereas the Laplace equation has no real characteristics. Of course, one can factorize the Laplace operator by introducing the complex variable $\zeta = iy$, but this trick is just a conversion of the problem on integration to an equivalent problem on extracting the real part of the complex solution

$$u = f(x+iy) + g(x-iy).$$

Let us consider the hyperbolic equations written, using the characteristic variables, in the standard form

$$u_{xy} + a(x,y)u_x + b(x,y)u_y + c(x,y)u = 0. \tag{1.5}$$

Can we factorize the left-hand side of every equation (1.5), i.e. write it as the product of two first-order linear differential operators? Leonard Euler [19] (see also [95], Introduction, and the references therein) showed that, in general, the answer to this question is negative. Furthermore, he gave the necessary and sufficient condition for Eq. (1.5) to be factorable and formulated the result in terms of the quantities

$$h = a_x + ab - c, \quad k = b_y + ab - c. \tag{1.6}$$

Namely, he demonstrated that Eq. (1.5) is factorable if and only if at least one of the quantities (1.6) vanishes. The solution of the factorized equation (1.5) is obtained by the consecutive integration of two first-order ordinary differential equations. If $h = k = 0$, Eq. (1.5) is reducible to the wave equation (1.4).

Later, the quantities (1.6) were rediscovered by Laplace [75] in connection with what is called today "Laplace's cascade method" and became known in the literature as the *Laplace invariants.*

In order to understand if Euler's method strongly requires *two families of characteristics* or it can be extended to parabolic equations using only *one family of characteristics*, I first investigated in [41] the question on existence of invariants of parabolic equations similar the Laplace invariants.

The present paper (see also [53]) is a continuation of the work [41] and gives an affirmative answer to the question on possibility of an extension of Euler's method to the parabolic equations. First of all, the linear parabolic equations with two independent variables are transformed to the *standard form*

$$u_t - u_{xx} + a(t,x)u_x + c(t,x)u = 0 \tag{1.7}$$

by an appropriate change of coordinates. Then the condition of reducibility of Eq. (1.7) to the heat equation (1.2) is obtained in terms of the invariant

$$K = aa_x - a_{xx} + a_t + 2c_x \tag{1.8}$$

of Eq. (1.7) found in [41]. Namely, it is shown that Eq. (1.7) can be mapped to the heat equation (1.2) by an appropriate change of the dependent variable if and only if the invariant (1.8) vanishes, i.e. $K = 0$.

The method developed here allows one to derive an explicit formula for the general solution of a wide class of parabolic equations. In particular, the general solution of the Black-Scholes equation is obtained and used for the solution of the Cauchy problem.

2 Euler's method of integration of hyperbolic equations

2.1 Standard form of hyperbolic equations

The general form of the homogeneous linear second-order partial differential equations with two independent variables, x and y, is

$$A\,u_{xx} + 2B\,u_{xy} + C\,u_{yy} + a\,u_x + b\,u_y + c\,u = 0, \tag{2.1}$$

where $A = A(x, y), \ldots, c = c(x, y)$ are prescribed functions. The terms with the second derivatives,

$$A\,u_{xx} + 2B\,u_{xy} + C\,u_{yy}, \tag{2.2}$$

compose the *principal part* of Equation (2.1).

The crucial step in studying Eq. (2.1) is the reduction of its principal part (2.2) to the so-called *standard form* by a change of variables

$$\xi = \varphi(x, y), \quad \eta = \psi(x, y). \tag{2.3}$$

Let us obtain the standard forms of the principal parts for Eq. (2.1). The change of variables (2.3) leads to the following transformation of derivatives:

$$\begin{aligned}
&u_x = \varphi_x u_\xi + \psi_x u_\eta, \qquad u_y = \varphi_y u_\xi + \psi_y u_\eta,\\
&u_{xx} = \varphi_x^2 u_{\xi\xi} + 2\varphi_x\psi_x u_{\xi\eta} + \psi_x^2 u_{\eta\eta} + \varphi_{xx} u_\xi + \psi_{xx} u_\eta,\\
&u_{yy} = \varphi_y^2 u_{\xi\xi} + 2\varphi_y\psi_y u_{\xi\eta} + \psi_y^2 u_{\eta\eta} + \varphi_{yy} u_\xi + \psi_{yy} u_\eta,\\
&u_{xy} = \varphi_x\varphi_y u_{\xi\xi} + (\varphi_x\psi_y + \varphi_y\psi_x) u_{\xi\eta} + \psi_x\psi_y u_{\eta\eta} + \varphi_{xy} u_\xi + \psi_{xy} u_\eta.
\end{aligned} \tag{2.4}$$

Substituting the expressions (2.4) in (2.1) we see that Eq. (2.1) is written in the new variables as follows:

$$\widetilde{A}\,u_{\xi\xi} + 2\widetilde{B}\,u_{\xi\eta} + \widetilde{C}\,u_{\eta\eta} + \tilde{a} u_\xi + \tilde{b} u_\eta + \tilde{c} u = 0, \tag{2.5}$$

where

$$\begin{aligned}
&\widetilde{A} = A\varphi_x^2 + 2B\varphi_x\varphi_y + C\varphi_y^2,\\
&\widetilde{B} = A\varphi_x\psi_x + B(\varphi_x\psi_y + \varphi_y\psi_x) + C\varphi_y\psi_y,\\
&\widetilde{C} = A\psi_x^2 + 2B\psi_x\psi_y + C\psi_y^2\\
&\tilde{a} = A\varphi_{xx} + 2B\varphi_{xy} + C\varphi_{yy} + a\varphi_x + b\varphi_y,\\
&\tilde{b} = A\psi_{xx} + 2B\psi_{xy} + C\psi_{yy} + a\psi_x + b\psi_y,\\
&\tilde{c} = c.
\end{aligned} \tag{2.6}$$

The principal part of Eq. (2.5) is

$$\widetilde{A}\,u_{\xi\xi} + 2\widetilde{B}\,u_{\xi\eta} + \widetilde{C}\,u_{\eta\eta}. \tag{2.7}$$

It is manifest from (2.6) that the principal part (2.7) will have only one term, $2\widetilde{B}\,u_{\xi\eta}$, if we choose for $\varphi(x,y)$ and $\psi(x,y)$ two functionally independent solutions,
$\omega_1 = \varphi(x,y)$ and $\omega_2 = \psi(x,y)$, of the equation

$$A\,\omega_x^2 + 2B\,\omega_x\omega_y + C\,\omega_y^2 = 0 \tag{2.8}$$

known as the *characteristic equation* for Eq. (2.1). Recall that for hyperbolic equations (2.1) the characteristic equation (2.8) has precisely two functionally independent solutions.

If $\omega(x,y)$ is any solution of Eq. (2.1), the curves

$$\omega(x,y) = \text{const.} \tag{2.9}$$

are called *characteristics* of Eq. (2.1). In order to find the characteristics, we set

$$\frac{\omega_x}{\omega_y} = \lambda \tag{2.10}$$

and rewrite the characteristic equation (2.8) in the form

$$A(x,y)\lambda^2 + 2B(x,y)\lambda + C(x,y) = 0. \tag{2.11}$$

For hyperbolic equations $B^2 - AC > 0$ and the quadratic equation (2.11) has two distinct real roots, $\lambda_1(x,y)$ and $\lambda_2(x,y)$ given by

$$\lambda_1(x,y) = \frac{-B + \sqrt{B^2 - AC}}{A}\,, \quad \lambda_2(x,y) = \frac{-B - \sqrt{B^2 - AC}}{A}\,\cdot \tag{2.12}$$

Substituting them in (2.10), we see that the characteristic equation (2.8) splits into two different linear first-order partial differential equations:

$$\frac{\partial\omega}{\partial x} - \lambda_1\,\frac{\partial\omega}{\partial y} = 0, \quad \frac{\partial\omega}{\partial x} - \lambda_2\,\frac{\partial\omega}{\partial y} = 0. \tag{2.13}$$

The characteristic systems for the equations (2.13) are

$$\frac{dx}{1} + \frac{dy}{\lambda_1(x,y)} = 0, \quad \frac{dx}{1} + \frac{dy}{\lambda_2(x,y)} = 0. \tag{2.14}$$

Each equation (2.14) has one independent first integral, $\varphi(x,y) = \text{const.}$ and $\psi(x,y) = \text{const.}$ for the first and the second equation (2.14), respectively.

Accordingly, the functions $\varphi(x,y)$ and $\psi(x,y)$ satisfy the first and the second equation (2.13), respectively:

$$\frac{\partial \varphi}{\partial x} - \lambda_1 \frac{\partial \varphi}{\partial y} = 0, \quad \frac{\partial \psi}{\partial x} - \lambda_2 \frac{\partial \psi}{\partial y} = 0, \tag{2.15}$$

and hence they are functionally independent. Thus, they provide two functionally independent solutions of the characteristic equation (2.11) and therefore one can take them as the right-hand sides in the change of variables (2.3). The new variables

$$\xi = \omega_1(x,y), \quad \eta = \omega_2(x,y), \tag{2.16}$$

where $\omega_1(x,y)$ and $\omega_2(x,y)$ are two functionally independent solutions of the characteristic equation are termed the *characteristic variables.* Thus, in the characteristic variables Eq. (2.5) becomes

$$2\widetilde{B}\, u_{\xi\eta} + \tilde{a} u_\xi + \tilde{b} u_\eta + \tilde{c} u = 0.$$

Dividing it by $2\widetilde{B}$, skipping the tilde and denoting ξ and η by x and y, respectively, we arrive at the following *standard form* of the hyperbolic equations:

$$u_{xy} + a(x,y)u_x + b(x,y)u_y + c(x,y)u = 0. \tag{2.17}$$

2.2 Essence of Euler's method

We owe to Leonard Euler [19] the first significant results in integration theory of hyperbolic equations. He generalized d'Alembert's solution to a wide class of Eqs. (2.17). He introduced the quantities*

$$h = a_x + ab - c, \quad k = b_y + ab - c \tag{2.18}$$

and showed that Eq. (2.17) is factorable if and only if at least one of the quantities h and k vanishes (see [19]; see also [95], Introduction, and the references therein). The solution of the factorized equation (2.17) is obtained by the consecutive integration of two first-order ordinary differential equations.

Euler's method consists in the following. Consider Eq. (2.17) with $h = 0$. Then this equation is factorable in the form

$$\left(\frac{\partial}{\partial x} + b\right)\left(\frac{\partial u}{\partial y} + au\right) = 0. \tag{2.19}$$

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*The quantities (2.18) were rediscovered by Laplace [75] in connection with what is called today "Laplace's cascade method" and became known in the literature as the *Laplace invariants.*

Setting

$$v = u_y + a\,u \tag{2.20}$$

one can rewrite Equation (2.19) as a first-order equation

$$v_x + b\,v = 0$$

and integrate it to obtain:

$$v = Q(y)\mathrm{e}^{-\int b(x,y)dx}. \tag{2.21}$$

Now we substitute (2.21) in (2.20), integrate the resulting non-homogeneous linear equation

$$u_y + au = Q(y)\mathrm{e}^{-\int b(x,y)dx} \tag{2.22}$$

with respect to y and obtain the following general solution of Eq. (2.17) with $h = 0$:

$$u = \Big[P(x) + \int Q(y)\mathrm{e}^{\int a\,dy - b\,dx}dy\Big]\mathrm{e}^{-\int a\,dy}, \tag{2.23}$$

where $P(x)$ and $Q(y)$ are arbitrary functions.

Likewise, if $k = 0$, Eq. (2.17) is factorable in the form

$$\left(\frac{\partial}{\partial y} + a\right)\left(\frac{\partial u}{\partial x} + b\,u\right) = 0. \tag{2.24}$$

In this case, we replace the substitution (2.20) by

$$w = u_x + bu \tag{2.25}$$

Now we repeat the calculations made in the case $h = 0$ and obtain the following general solution of Eq. (2.17) with $k = 0$:

$$u = \left[Q(y) + \int P(x)e^{\int b\,dx - a\,dy}dx\right] e^{-\int b\,dx}. \tag{2.26}$$

2.3 Equivalence transformations

In particular, Euler's method allows one to identify those equations (2.17) that can be reduced to the wave equation by a change of variables, and hence, solved by d'Alembert's method. We will single out all such equations in the next section. Here we discuss the most general form of the changes of variables preserving the linearity and homogeneity of hyperbolic equations as well as their standard form (2.17). These changes of variables are termed *equivalence transformations.* They are well known and have the form

$$\widetilde{x} = f(x), \quad \widetilde{y} = g(y), \quad v = \sigma(x,y)\,u, \tag{2.27}$$

where $f'(x) \neq 0,\ g'(y) \neq 0$, and $\sigma(x,y) \neq 0$. Here u and v are regarded as functions of x, y and $\widetilde{x}, \widetilde{y}$, respectively. The equations (2.17) related by an equivalence transformation (2.27) are said to be *equivalent.*

Let us begin with the restricted equivalence transformations (2.27) by setting
$\widetilde{x} = x,\ \widetilde{y} = y$ and find the equations (2.17) reducible to the wave equation by the linear transformation of the dependent variable written in the form

$$v = u\,\mathrm{e}^{\varrho(x,y)}. \tag{2.28}$$

We substitute the expressions

$$\begin{aligned} u &= v\,\mathrm{e}^{-\varrho(x,y)},\\ u_x &= (v_x - v\varrho_x)\,\mathrm{e}^{-\varrho(x,y)}, \quad u_y = (v_y - v\varrho_y)\,\mathrm{e}^{-\varrho(x,y)},\\ u_{xy} &= (v_{xy} - v_x\varrho_y - v_y\varrho_x - v\,\varrho_{xy} + v\,\varrho_x\,\varrho_y)\,\mathrm{e}^{-\varrho(x,y)} \end{aligned}$$

in the left-hand side of Eq. (2.17) and obtain:

$$\begin{aligned} &u_{xy} + a\,u_x + b\,u_y + cu\\ &= \big[v_{xy} + (a - \varrho_y)\,v_x + (b - \varrho_x)\,v_y\\ &+ (-\varrho_{xy} + \varrho_x\,\varrho_y - a\varrho_x - b\varrho_y + c)\,v\big]\mathrm{e}^{-\varrho(x,y)}. \end{aligned} \tag{2.29}$$

2.4 Reduction to the wave equation

Eq. (2.29) reduces to the wave equation $v_{xy} = 0$ if and only if the equations

$$a - \varrho_y = 0, \quad b - \varrho_x = 0 \tag{2.30}$$

and

$$\varrho_{xy} - \varrho_x\,\varrho_y + a\varrho_x + b\varrho_y - c = 0 \tag{2.31}$$

hold. Eqs. (2.30) provide a system of *two* equations for *one* unknown function $\varrho(x,y)$ of two variables. Recall that a system of equations is called an *over-determined system* if it contains more equations than unknown functions to be determined by the system in question. Over-determined systems have solutions only if they satisfy certain *compatibility conditions.*

Thus, the system of equations (2.30) is over-determined. Its compatibility condition is obtained from the equation $\varrho_{xy} = \varrho_{yx}$ and has the form

$$a_x = b_y. \tag{2.32}$$

Eq. (2.31), upon using Eqs. (2.30) and (2.32), is written

$$a_x + ab - c = 0. \tag{2.33}$$

In terms of the quantities h and k defined by (2.18) the conditions (2.32) and (2.33) are written in the following symmetric form:

$$h = 0, \quad k = 0. \tag{2.34}$$

Remark 6.1. The equations (2.34) are invariant under the general equivalence transformation (2.27). In consequence, the change of the independent variables does not provide new equations reducible to the wave equation.

Summing up the above calculations and taking into account Remark 6.1, we arrive at the following result.

Theorem 6.1. Eq. (2.17) is equivalent to the wave equation if and only if Eqs. (2.34) are satisfied. Any equation (2.17) with $h = k = 0$ can be reduced to the wave equation $v_{xy} = 0$ by the linear transformation of the dependent variable:

$$u = v\,\mathrm{e}^{-\varrho(x,y)} \tag{2.35}$$

without changing the independent variables x and y. The function ϱ in (2.35) is obtained by solving the following compatible system:

$$\frac{\partial \varrho}{\partial x} = b(x,y), \quad \frac{\partial \varrho}{\partial y} = a(x,y). \tag{2.36}$$

Theorem 6.1 furnishes us with a practical method for solving a wide class of hyperbolic equations (2.1) by reducing them to the wave equation. In order to apply this method, one has to rewrite Eq. (2.1) in the standard form (2.17) by introducing the characteristic variables (2.16). Then one should calculate the Laplace invariants (2.18). If $h = k = 0$, one can find $\varrho(x,y)$ by solving the equations (2.36) and reduce the equation in question to the wave equation $v_{xy} = 0$ by the transformation (2.35). Finally, substituting

$$v = f(x) + g(y)$$

into (2.35) one will obtain the solution in the characteristic variables:

$$u = [f(x) + g(y)]\,\mathrm{e}^{-\varrho(x,y)}. \tag{2.37}$$

Example 6.1. Let us illustrate the method by the equation (see [47], Section 5.3.2))

$$\frac{u_{xx}}{x^2} - \frac{u_{yy}}{y^2} + 3\left(\frac{u_x}{x^3} - \frac{u_y}{y^3}\right) = 0. \tag{2.38}$$

Here Eq. (2.8) for the characteristics has the form

$$\left(\frac{\omega_x}{x}\right)^2 - \left(\frac{\omega_y}{y}\right)^2 = \left(\frac{\omega_x}{x} - \frac{\omega_y}{y}\right)\left(\frac{\omega_x}{x} + \frac{\omega_y}{y}\right) = 0.$$

It splits into two equations:

$$\frac{\omega_x}{x} + \frac{\omega_y}{y} = 0, \quad \frac{\omega_x}{x} - \frac{\omega_y}{y} = 0.$$

They have the following first integrals:

$$x^2 - y^2 = \text{const.}, \quad x^2 + y^2 = \text{const.}$$

Hence, the characteristic variables (2.16) are defined by

$$\xi = x^2 - y^2, \quad \eta = x^2 + y^2. \tag{2.39}$$

We have (cf. Eqs. (2.4)):

$$\begin{aligned} u_x &= u_\xi \cdot \xi_x + u_\eta \cdot \eta_x = 2x(u_\xi + u_\eta), \\ u_y &= u_\xi \cdot \xi_y + u_\eta \cdot \eta_y = 2y(u_\eta - u_\xi), \\ u_{xx} &= 2(u_\xi + u_\eta) + 4x^2[(u_\xi + u_\eta)_\xi + (u_\xi + u_\eta)_\eta] \\ &= 2(u_\xi + u_\eta) + 4x^2(u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}), \\ u_{yy} &= 2(u_\eta - u_\xi) + 4y^2[(u_\eta - u_\xi)_\eta - (u_\eta - u_\xi)_\xi] \\ &= 2(u_\eta - u_\xi) + 4y^2(u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta}). \end{aligned}$$

Therefore, Eq. (2.38) takes the following form:

$$u_{\xi\eta} + \frac{x^2 + y^2}{2x^2y^2}\, u_\xi - \frac{x^2 - y^2}{2x^2y^2}\, u_\eta = 0.$$

Invoking Eqs. (2.39) and noting that

$$\eta^2 - \xi^2 = 4x^2y^2, \tag{2.40}$$

we ultimately arrive at the following standard form (2.17) of Eq. (2.38):

$$u_{\xi\eta} + \frac{2\eta}{\eta^2 - \xi^2}\, u_\xi - \frac{2\xi}{\eta^2 - \xi^2}\, u_\eta = 0. \tag{2.41}$$

The coefficients of Eq. (2.41) are:

$$a = \frac{2\eta}{\eta^2 - \xi^2}\,, \quad b = -\frac{2\xi}{\eta^2 - \xi^2}\,, \quad c = 0.$$

We substitute them in (2.18) where we replace x and y by ξ and η, respectively. We have

$$a_\xi = b_\eta = \frac{4\xi\eta}{(\eta^2 - \xi^2)^2}$$

and see that $h = k = 0$. Now we solve Eqs. (2.36):

$$\frac{\partial \varrho}{\partial \eta} = \frac{2\eta}{\eta^2 - \xi^2}, \quad \frac{\partial \varrho}{\partial \xi} = -\frac{2\xi}{\eta^2 - \xi^2}$$

and obtain

$$\varrho = \ln(\eta^2 - \xi^2).$$

Hence, the substitution (2.35) is written

$$v = u e^{\ln(\eta^2 - \xi^2)} = (\eta^2 - \xi^2)\, u. \tag{2.42}$$

It maps Eq. (2.41) to the wave equation

$$v_{\xi\eta} = 0. \tag{2.43}$$

Therefore

$$v(\xi, \eta) = f(\xi) + g(\eta),$$

and (2.41) yields:

$$u(\xi, \eta) = \frac{f(\xi) + h(\eta)}{\eta^2 - \xi^2}\,\cdot$$

Returning to the original variables by using Eqs. (2.39) and denoting $F = f/4$ and $H = h/4$, we finally obtain the following general solution to Eq. (2.38):

$$u(x, y) = \frac{F(x^2 - y^2) + H(x^2 + y^2)}{x^2 y^2}\,\cdot \tag{2.44}$$

Remark 6.2. We can integrate Eq. (2.38) by rewriting it in a factorized form in the original variables. Note that the differentiations D_x, D_y in the original variables are related with the differentiations D_ξ, D_η in the new variables (2.3) as follows (see (2.4)):

$$D_x = \xi_x D_\xi + \eta_x D_\eta, \quad D_y = \xi_y D_\xi + \eta_y D_\eta, \tag{2.45}$$

whence

$$D_\xi = \frac{1}{\Omega}\left(\eta_y D_x - \eta_x D_y\right), \quad D_\eta = \frac{1}{\Omega}\left(\xi_x D_y - \xi_y D_x\right) \tag{2.46}$$

with

$$\Omega = \xi_x \eta_y - \xi_y \eta_x.$$

Applying Eqs. (2.46) to the change of variables (2.39) we obtain:

$$D_\xi = \frac{1}{4}\left(\frac{1}{x} D_x - \frac{1}{y} D_y\right), \quad D_\eta = \frac{1}{4}\left(\frac{1}{x} D_x + \frac{1}{y} D_y\right).$$

Therefore, Eq. (2.43) written in the form

$$D_\xi D_\eta(v) = 0$$

and Eqs. (2.42), (2.40) yield the following factorization of Eq. (2.38):

$$L_1 L_2(u) = 0, \tag{2.47}$$

where $L_1,\ L_2$ are the first-order linear differential operators given by

$$L_1 = \frac{1}{x}D_x - \frac{1}{y}D_y, \quad L_2 = \left(\frac{1}{x}D_x + \frac{1}{y}D_y\right)x^2y^2. \tag{2.48}$$

To solve Eq. (2.47), we denote $L_2(u) = w$, write Eq. (2.47) as the the first-order equation

$$L_1(w) \equiv \frac{1}{x}\frac{\partial w}{\partial x} - \frac{1}{y}\frac{\partial w}{\partial y} = 0$$

for w, whence

$$w = \phi(\eta), \quad \eta = x^2 + y^2,$$

on integration. It remains to integrate the non-homogeneous first-order equation
$L_2(u) = \phi(\eta)$:

$$\left(\frac{1}{x}\frac{\partial}{\partial x} + \frac{1}{y}\frac{\partial}{\partial y}\right)(x^2y^2u) = \phi(\eta).$$

Denoting $x^2y^2u = v$ for the sake of simplicity, we rewrite the equation in the form

$$\frac{1}{x}\frac{\partial v}{\partial x} + \frac{1}{y}\frac{\partial v}{\partial y} = \phi(\eta).$$

The first equation of the characteristic system

$$xdx = ydy = \frac{dv}{\phi(\eta)}$$

provides the first integral $x^2 - y^2 \equiv \xi = \text{const}$. Substituting $\eta = 2x^2 + \xi$ in the second equation of the characteristic system, $dv = x\phi(\eta)dx$ we have:

$$dv = \phi(\xi + 2x^2)xdx = \frac{1}{4}\,\phi(\xi + 2x^2)d(x\xi + 2x^2),$$

whence

$$v = H(\xi + 2x^2) + F(\xi).$$

Since $\xi + 2x^2 = \eta$ and $v = x^2y^2u$, we finally obtain

$$x^2y^2u = H(\eta) + F(\xi),$$

i.e. the solution (2.44):

$$u = \frac{H(x^2 + y^2) + F(x^2 - y^2)}{x^2 y^2} \cdot$$

It is manifest from the above calculations that the use of characteristic variables simplifies the integration procedure significantly.

3 Parabolic equations

3.1 Equivalence transformations

The standard form of the parabolic equations is

$$u_t + A(t,x)u_{xx} + a(t,x)u_x + c(t,x)u = 0. \tag{3.1}$$

The equivalence transformations of Eqs. (3.1) comprise the invertible changes of the independent variables of the form

$$\tau = \phi(t), \quad y = \psi(t,x) \tag{3.2}$$

and the linear transformation of the dependent variable

$$v = \sigma(t,x)u, \quad \sigma(t,x) \neq 0, \tag{3.3}$$

where $\phi(t), \psi(t,x)$ and $\sigma(t,x)$ are arbitrary functions.

Under the change of the independent variables (3.2) the derivatives of u undergo the following transformations:

$$u_t = \phi' u_\tau + \psi_t u_y, \quad u_x = \psi_x u_y, \quad u_{xx} = \psi_x^2 u_{yy} + \psi_{xx} u_y,$$

and Eq. (3.1) becomes:

$$\phi' u_\tau + A\psi_x^2 u_{yy} + \big(\psi_t + A\psi_{xx} + a\psi_x\big)u_y + cu = 0.$$

Therefore, by choosing ψ such that $|A|\psi_x^2 = 1$ and taking $\phi = \pm\, t$ in accordance with the sign of A, we can rewrite any parabolic equation in the form

$$u_t - u_{xx} + a(t,x)u_x + c(t,x)u = 0. \tag{3.4}$$

In what follows, we will use the parabolic equations (3.4) and employ their equivalence transformation (3.3) written in the form (cf. Eq. (2.28)):

$$v = u\, \mathrm{e}^{\varrho(t,x)}. \tag{3.5}$$

Solving Eq. (3.5) for u and differentiating, we have:

$$\begin{aligned} u &= v\,\mathrm{e}^{-\varrho(t,x)}, \quad u_t = (v_t - v\varrho_t)\,\mathrm{e}^{-\varrho(t,x)},\\ u_x &= (v_x - v\varrho_x)\,\mathrm{e}^{-\varrho(t,x)},\\ u_{xx} &= \big[v_{xx} - 2v_x\varrho_x + (\varrho_x^2 - \varrho_{xx})v\big]\,\mathrm{e}^{-\varrho(t,x)}. \end{aligned} \tag{3.6}$$

Inserting the expressions (3.6) in the left-hand side of Eq. (3.4), we obtain:

$$\begin{aligned} & u_t - u_{xx} + au_x + cu\\ & = \big[v_t - v_{xx} + (a + 2\varrho_x)\,v_x\\ & + (\varrho_{xx} - \varrho_x^2 - \varrho_t - a\varrho_x + c)\,v\big]\mathrm{e}^{-\varrho(t,x)}. \end{aligned} \tag{3.7}$$

3.2 Semi-invariant. Equations reducible to the heat equation

Eq. (3.7) shows that Eq. (3.4) can be reduced to the heat equation

$$v_t - v_{xx} = 0, \quad t > 0, \tag{3.8}$$

by an equivalence transformation (3.5) if and only if

$$a + 2\varrho_x = 0, \quad \varrho_{xx} - \varrho_x^2 - \varrho_t - a\varrho_x + c = 0. \tag{3.9}$$

The first equation (3.9) yields

$$\varrho_x = -\frac{1}{2}\,a, \quad \varrho_{xx} = -\frac{1}{2}\,a_x,$$

and hence the second equation (3.9) becomes

$$\frac{1}{4}\,a^2 - \frac{1}{2}\,a_x - \varrho_t + c = 0.$$

Thus, Eqs. (3.9) can be rewritten as the following over-determine system of first-order equations for the unknown function $\varrho(t, x)$:

$$\varrho_x = -\frac{1}{2}\,a, \quad \varrho_t = \frac{1}{4}\,a^2 - \frac{1}{2}\,a_x + c. \tag{3.10}$$

The compatibility condition $\varrho_{xt} = \varrho_{tx}$ for the system (3.10) has the form

$$aa_x - a_{xx} + a_t + 2c_x = 0. \tag{3.11}$$

The left-hand side of Eq. (3.11) is the invariant (more specifically, *semi-invariant*) for the parabolic equations first obtained by the author in 2000

(Preprint) and published in [41]. Namely, it is shown in [41] that the linear parabolic equations

$$u_t + A(t,x)u_{xx} + B(t,x)u_x + C(t,x)u = 0$$

have the following invariant with respect to the equivalence transformation (3.5):

$$K = \frac{1}{2}\,B^2A_x + \Big(A_t + AA_{xx} - A_x^2\Big)B + (AA_x - AB)B_x - AB_t - A^2B_{xx} + 2A^2C_x.$$

Since Eq. (3.4) corresponds to $A = -1,\ B = a,\ C = c$, the above semi-invariant is written

$$K = aa_x - a_{xx} + a_t + 2c_x. \tag{3.12}$$

Summing up the above calculations, we arrive at the following result.

Theorem 6.2. The parabolic equation (3.4)

$$u_t - u_{xx} + a(t,x)u_x + c(t,x)u = 0$$

can be reduced to the heat equation (3.8)

$$v_t - v_{xx} = 0$$

by an appropriate linear transformation (3.5) of the dependent variable,

$$u = v\,\mathrm{e}^{-\varrho(t,x)}, \tag{3.13}$$

without changing the independent variables t and x if and only if the semi-invariant (3.12) vanishes, $K = 0$. The function ϱ in the transformation (3.5) of Eq. (3.4) to the heat equation is obtained by solving the following compatible system (3.10):

$$\frac{\partial\varrho}{\partial x} = -\frac{1}{2}\,a, \quad \frac{\partial\varrho}{\partial t} = \frac{1}{4}\,a^2 - \frac{1}{2}\,a_x + c.$$

The system (3.10) is compatible (has a solution $\varrho(t,x)$) due to $K = 0$, i.e. Eq. (3.11):

$$aa_x - a_{xx} + a_t + 2c_x = 0.$$

Corollary 6.1. Any Eq. (3.4) with constant coefficients a and c can be reduced to the heat equation. Indeed, the semi-invariant (3.12) vanishes if $a, c = \text{const}$.

Remark 6.3. For hyperbolic equations a similar statement does not valid. For example, the telegraph equation $u_{xy} + u = 0$ cannot be reduced to the wave equation.

Example 6.2. ([76], [41]). The semi-invariant (3.12) of any equation of the form

$$u_t - u_{xx} + c(t)u = 0$$

vanishes. Therefore this equation reduces to the heat equation by the equivalence transformation $u = v\,\mathrm{e}^{\int c(t)dt}$.

Theorem 6.2 furnishes us with a practical method for solving by quadrature a wide class of parabolic equations (3.4) by reducing them to the heat equation. First, let us discuss the known solutions to the heat equation.

3.3 Poisson's solution

Recall that, given any continuous and bounded function $f(x)$, the function $v(t,x)$ defined by Poisson's formula

$$v(t,x) = \frac{1}{2\sqrt{\pi t}} \int\limits_{-\infty}^{+\infty} f(z)\,\mathrm{e}^{-\frac{(x-z)^2}{4t}}\,dz, \quad t > 0, \tag{3.14}$$

solves the heat equation (3.8) and satisfies the initial condition

$$v(0,x) \equiv \lim_{t\to+0} v(t,x) = f(x). \tag{3.15}$$

To make the text self-contained, let us verify that the function $v(t,x)$ defined by (3.14) satisfies the heat equation. The integral (3.14) itself as well as the integrals obtained by differentiating it under the integral sign any number of times converge uniformly due to the presence of the rapidly decreasing factor $\mathrm{e}^{-\frac{(x-z)^2}{4t}}$ when $t > 0$. The integral (3.14) solves the heat equation $v_t = v_{xx}$ because

$$v_t = -\frac{1}{4t\sqrt{\pi t}} \int\limits_{-\infty}^{+\infty} f(z)\,\mathrm{e}^{-\frac{(x-z)^2}{4t}}\,dz + \frac{1}{8t^2\sqrt{\pi t}} \int\limits_{-\infty}^{+\infty} (x-z)^2 f(z)\,\mathrm{e}^{-\frac{(x-z)^2}{4t}}\,dz,$$

$$v_x = -\frac{1}{4t\sqrt{\pi t}} \int\limits_{-\infty}^{+\infty} (x-z) f(z)\,\mathrm{e}^{-\frac{(x-z)^2}{4t}}\,dz,$$

$$v_{xx} = -\frac{1}{4t\sqrt{\pi t}} \int\limits_{-\infty}^{+\infty} f(z)\,\mathrm{e}^{-\frac{(x-z)^2}{4t}}\,dz + \frac{1}{8t^2\sqrt{\pi t}} \int\limits_{-\infty}^{+\infty} (x-z)^2 f(z)\,\mathrm{e}^{-\frac{(x-z)^2}{4t}}\,dz.$$

3.4 Uniqueness class and the general solution therein

The uniqueness of the solution to the Cauchy problem guarantees that (3.14) provides the *general solution* to the heat equation in the half-plane $t > 0$ provided that the solution is bounded.

In general, the solution to the Cauchy problem for the heat equation may not be unique in the class of unbounded functions. It can be shown, however, that the presence of the factor $\mathrm{e}^{-\frac{(x-z)^2}{4t}}$ under the integral sign in (3.14) guarantees the uniqueness of the solution in the class of continuous functions $v(t,x)$ defined on a strip

$$0 \le t \le T < +\infty, \ -\infty < x < +\infty$$

and such that $|v(t,x)|$ grows not faster than e^{x^2} as $x \to \infty$. In other words, the following statement holds (see, e.g. [96], Ch. IV).

Theorem 6.3. The solution to the Cauchy problem

$$v_t - v_{xx} = 0 \ (t > 0), \quad v(0,x) \equiv \lim_{t\to+0} v(t,x) = f(x)$$

is unique and is given by Poisson's formula (3.14) in the class of functions $v(t,x)$ satisfying the following condition:

$$\max_{0\le t\le T} \big|v(t,x)\big| \,\mathrm{e}^{-\beta x^2} \to 0 \quad \text{as} \quad |x| \to \infty, \tag{3.16}$$

where β is a certain constant.

Corollary 6.2. Let $\tilde{v}(t,x)$ be a continuous function satisfying the uniqueness condition (3.16). If $\tilde{v}(t,x)$ solves the heat equation for $t > 0$, then it is given by Poisson's formula (3.14) with a certain function $f(x)$.

Proof. Substituting in the integral (3.14) the function $f(x) = \tilde{v}(0,x)$, we obtain a solution $v(t,x)$ of the heat equation. Since the initial values of both solutions $\tilde{v}(t,x)$ and $v(t,x)$ coincide, $v(0,x) = f(x) = \tilde{v}(0,x)$, the uniqueness of the solution to the Cauchy problem shows that $\tilde{v}(t,x) = v(t,x)$, i.e. that the solution $\tilde{v}(t,x)$ is given by the formula (3.14).

According to Corollary 6.2, all continuous solutions of the heat equation (3.8) satisfying the condition (3.16) admit the integral representation (3.14). In other words, *Poisson's formula* (3.14) *provides the general solution of the heat equation in the class of functions satisfying the condition* (3.16).

3.5 Tikhonov's solution

A.N. Tikhonov [103] showed that the condition (3.16) is exact. Namely, he noticed that there are solutions of the heat equation that are not identically zero but vanish at $t = 0$,

$$v(0, x) = 0, \quad -\infty < x < +\infty,$$

and satisfy the following condition with any small $\varepsilon > 0$:

$$\max_{0 \le t \le T} \left| v(t, x) \right| \mathrm{e}^{-x^{2+\varepsilon}} \to 0 \quad \text{as} \quad |x| \to \infty.$$

These solutions are, in fact, particular cases of the following *Tikhonov's solution*:

$$\begin{aligned} v(t, x) =& F(t) + xF_1(t) + \frac{x^2}{2!}\, F'(t) + \frac{x^3}{3!}\, F_1'(t) + \cdots \\ &+ \frac{x^{2n}}{(2n)!}\, F^{(n)}(t) + \frac{x^{2n+1}}{(2n+1)!}\, F_1^{(n)}(t) + \cdots \end{aligned} \tag{3.17}$$

given in [103]. Here $F(t)$ and $F_1(t)$ are any two C^∞ functions (i.e. differentiable infinitely many times) such that the series (3.17) is uniformly convergent. Let us verify that the series (3.17) solves the heat equation (3.8). We can differentiate each term of the series (3.17) due to its uniform convergence and obtain:

$$\begin{aligned} v_t =& F'(t) + xF_1'(t) + \frac{x^2}{2!}\, F''(t) + \frac{x^3}{3!}\, F_1''(t) + \cdots \\ &+ \frac{x^{2n}}{(2n)!}\, F^{(n+1)}(t) + \frac{x^{2n+1}}{(2n+1)!}\, F_1^{(n+1)}(t) + \cdots\,, \\ v_x =& F_1(t) + xF'(t) + \frac{x^2}{2!}\, F_1'(t) + \frac{x^3}{3!}\, F''(t) + \frac{x^4}{4!}\, F_1''(t) + \cdots \\ &+ \frac{x^{2n+1}}{(2n+1)!}\, F^{(n+1)}(t) + \frac{x^{2n+2}}{(2n+2)!}\, F_1^{(n+1)}(t) + \cdots\,, \\ v_{xx} =& F'(t) + x\, F_1'(t) + \frac{x^2}{2!}\, F''(t) + \frac{x^3}{3!}\, F_1''(t) + \cdots \\ &+ \frac{x^{2n}}{(2n)!}\, F^{(n+1)}(t) + \frac{x^{2n+1}}{(2n+1)!}\, F_1^{(n+1)}(t) + \cdots\,. \end{aligned}$$

Subtracting term by term we obtain $v_t - v_{xx} = 0$.

Tikhonov also showed that any solution $v(t,x)$ of the heat equation defined for all x and $t > 0$ can be represented in the form (3.17). This solution satisfies the conditions

$$v(t,0) = F(t), \quad v_x(t,0) = F_1(t). \tag{3.18}$$

Furthermore, the solution of the heat equation satisfying the conditions (3.18) is unique.

Note that the solution (3.17) is the superposition of two different solutions:

$$v(t,x) = F(t) + \frac{x^2}{2!}\, F'(t) + \frac{x^4}{4!}\, F''(t) + \cdots + \frac{x^{2n}}{(2n)!}\, F^{(n)}(t) + \cdots \tag{3.19}$$

and

$$v(t,x) = xF_1(t) + \frac{x^3}{3!}\, F_1'(t) + \frac{x^5}{5!}\, F_1''(t) + \cdots + \frac{x^{2n+1}}{(2n+1)!}\, F_1^{(n)}(t) + \cdots . \tag{3.20}$$

The infinite series representations (3.17), (3.19) and (3.20) of solutions to the heat equation are particularly useful for obtaining approximate solutions to the heat equation (3.8) and to the equivalent equations, e.g. by truncating the infinite series. Tikhonov's series representations are also convenient for obtaining solutions in closed forms, in particular, in terms of elementary functions. One of such cases is obtained by taking for $F(t)$ and $F_1(t)$ any polynomials. Let us consider examples.

Example 6.3. Letting in (3.20)

$$F_1(t) = a + bt + ct^2 + kt^3,$$

we obtain the following polynomial solution:

$$v(t,x) = (a + bt + ct^2 + kt^3)x + \frac{1}{3!}\,\big(b + 2ct + 3kt^2\big)x^3 + \frac{1}{5!}\,\big(2c + 6kt\big)x^5 + \frac{6k}{7!}\,x^7.$$

In particular, taking $a = b = k = 0$ and $c = 60$, we obtain the solution

$$v(t,x) = 60t^2x + 20tx^3 + x^5. \tag{3.21}$$

The solution (3.21) satisfies the initial condition $v(0,x) = x^5$, and hence admits the following integral representation (3.14) when $t > 0$:

$$v(t,x) = \frac{1}{2\sqrt{\pi t}} \int\limits_{-\infty}^{+\infty} z^5\, \mathrm{e}^{-\frac{(x-z)^2}{4t}}\, dz. \tag{3.22}$$

It is manifest that that the explicit form (3.21) of the solution is essentially simpler.

Example 6.4. Setting $F(t) = \mathrm{e}^{-t}$ and $F_1(t) = \mathrm{e}^{-t}$ in (3.19) and (3.20), respectively, we obtain the following two particular solutions:

$$v(t,x) = \mathrm{e}^{-t}\cos x, \quad v(t,x) = \mathrm{e}^{-t}\sin x.$$

Let us discuss now applications of Tikhonov's solution (3.17) and Poisson's solution (3.14) to the parabolic equations (3.4) with the vanishing semi-invariant (3.12).

3.6 Integration of equations with the vanishing semi-invariant

The solution of parabolic equations Eq. (3.4) with the vanishing semi-invariant (3.12) is given by Eq. (3.13),

$$u = \mathrm{e}^{-\varrho(t,x)}\, v(t,x), \tag{3.13}$$

where the function $\varrho(t,x)$ is obtained by solving Eqs. (3.10), and $v(t,x)$ is the solution of the heat equation written either in Poisson's integral representation (3.14) or in Tikhonov's series representation (3.17). We will consider separately the use of Poisson's and Tikhonov's representations.

3.6.1 Poisson's form of the solution

In this section we will employ the uniqueness condition (3.16). Using Eqs. (3.13), (3.14) as well as Theorem 6.2 and Corollary 6.2, we arrive at the following statement.

Theorem 6.4. Let the semi-invariant (3.12) of Eq. (3.4) vanish. Then any solution to Eq. (3.4) belonging to the class of functions satisfying the condition (cf. (3.16))

$$\max_{0\le t\le T}\left|u(t,x)\,\mathrm{e}^{\varrho(t,x)}\right|\mathrm{e}^{-\beta x^2} \to 0 \quad \text{as} \quad |x|\to\infty, \tag{3.23}$$

where $\varrho(t,x)$ is found by solving the system (3.10), admits the integral representation

$$u(t,x) = \frac{1}{2\sqrt{\pi t}}\,\mathrm{e}^{-\varrho(t,x)}\int\limits_{-\infty}^{+\infty} f(z)\,\mathrm{e}^{-\frac{(x-z)^2}{4t}}\,dz, \quad t>0. \tag{3.24}$$

Thus, Eq. (3.24) furnishes the general solution to Eq. (3.4) with the vanishing semi-invariant K, provided that the condition (3.23) is satisfied.

Example 6.5. The equation

$$u_t - u_{xx} + 2u_x - u = 0 \tag{3.25}$$

has the vanishing semi-invariant (3.12) (see Corollary 6.1). The system (3.10) yields

$$\varrho(t,x) = -x,$$

and hence Eq. (3.13) is written

$$u(t,x) = \mathrm{e}^x\, v(t,x). \tag{3.26}$$

Taking one of the simplest solutions to the heat equation, namely $v = x$, we get the solution

$$u(t,x) = x\mathrm{e}^x$$

to Eq. (3.25). This particular solution is unbounded. However, it satisfies the condition (3.23), and hence admits the integral representation (3.24) with a certain function $f(z)$. How to find the corresponding function $f(z)$? Let us discuss this question in general.

Poisson's form of the solution is well suited for solving the initial value problem not only for the heat equation but also for all parabolic equations with the vanishing semi-invariant. The result is formulated in the next theorem. In particular, it gives the answer to the questions similar to that formulated in Example 6.5.

Theorem 6.5. Let (3.4) be an equation with the vanishing semi-invariant (3.12). Then the solution to the initial value problem

$$u_t - u_{xx} + a(t,x)u_x + c(t,x)u = 0, \quad u\big|_{t=0} = u_0(x) \tag{3.27}$$

is given by

$$u(t,x) = \frac{1}{2\sqrt{\pi t}}\,\mathrm{e}^{-\varrho(t,x)} \int\limits_{-\infty}^{+\infty} u_0(z)\,\mathrm{e}^{\varrho(0,z)}\mathrm{e}^{-\frac{(x-z)^2}{4t}}\,dz, \quad t > 0, \tag{3.28}$$

where $\varrho(t,x)$ solves the system (3.10).

Proof. Letting $t \to +0$ in Eq. (3.24), using the initial condition

$$\lim_{t\to+0} u(t,x) = u\big|_{t=0} = u_0(x)$$

and the well-known equation

$$\lim_{t\to+0}\frac{1}{2\sqrt{\pi t}}\int_{-\infty}^{+\infty} f(z)\,\mathrm{e}^{-\frac{(x-z)^2}{4t}}\,dz = f(x), \tag{3.29}$$

we obtain $u_0(x) = \mathrm{e}^{-\varrho(0,x)} f(x)$. Hence, the solution to the problem (3.27) is given by Eq. (3.24) with

$$f(z) = u_0(z)\,\mathrm{e}^{\varrho(0,z)},$$

i.e. by Eq. (3.28).

Example 6.6. Let us give the answer to the question put in Example 6.5. Since $\varrho(t,x) = -x$ and the particular solution is given by $u(t,x) = x\mathrm{e}^x$, we have:

$$f(z) = u_0(z)\,\mathrm{e}^{\varrho(0,z)} = z\mathrm{e}^z\mathrm{e}^{-z} = z.$$

Therefore Eq. (3.28) provides the following integral representation (3.24) of the particular solution $u(t,x) = x\mathrm{e}^x$ to the equation $u_t - u_{xx} + 2u_x - u = 0$:

$$u(t,x) \equiv x\mathrm{e}^x = \frac{\mathrm{e}^x}{2\sqrt{\pi t}}\int_{-\infty}^{+\infty} z\,\mathrm{e}^{-\frac{(x-z)^2}{4t}}\,dz, \quad t > 0.$$

Example 6.7. Consider again Eq. (3.25),

$$u_t - u_{xx} + 2u_x - u = 0.$$

Substituting in Eq. (3.26),

$$u(t,x) = \mathrm{e}^x\,v(t,x),$$

the fundamental solution

$$v(t,x) = \frac{\theta(t)}{2\sqrt{\pi t}}\,\mathrm{e}^{-\frac{x^2}{4t}}$$

of the heat equation, we obtain $u = \mathcal{E}(t,x)$, where

$$\mathcal{E}(t,x) = \frac{\theta(t)}{2\sqrt{\pi t}}\,\mathrm{e}^{x-\frac{x^2}{4t}} \tag{3.30}$$

and $\theta(t)$ is the Heaviside function. The function (3.30) is the fundamental solution for Eq. (3.25). Indeed, writing the left-hand side of Eq. (3.25) in the operator form

$$L(u) = u_t - u_{xx} + 2u_x - u,$$

we see that the linear differential operator L acts on the function (3.30) as follows:

$$L(\mathcal{E}) = \frac{\theta'(t)}{2\sqrt{\pi t}}\,\mathrm{e}^{x-\frac{x^2}{4t}} + \theta(t)\,L\Big(\frac{1}{2\sqrt{\pi t}}\,\mathrm{e}^{x-\frac{x^2}{4t}}\Big).$$

Invoking that

$$L\Big(\frac{1}{2\sqrt{\pi t}}\,\mathrm{e}^{x-\frac{x^2}{4t}}\Big), \quad t > 0,$$

and

$$\theta'(t) = \delta(t), \quad F(t,x)\delta(t) = F(0,x)\delta(t), \quad \lim_{t\to+0}\frac{1}{2\sqrt{\pi t}}\,\mathrm{e}^{-\frac{x^2}{4t}} = \delta(x),$$

where δ is Dirac's δ-function, we obtain:

$$L(\mathcal{E}) = \mathrm{e}^{x}\,\delta(t)\,\lim_{t\to+0}\frac{1}{2\sqrt{\pi t}}\,\mathrm{e}^{-\frac{x^2}{4t}} = \mathrm{e}^{x}\,\delta(t)\delta(x).$$

Since $f(x)\delta(x) = f(0)\delta(x)$ for any C^∞-function, and $\delta(t)\delta(x) = \delta(t,x)$, we finally see that the function (3.30) satisfies the definition of the fundamental solution:

$$L(\mathcal{E}) = \delta(t,x).$$

3.6.2 Tikhonov's form of the solution

If we do not require the condition (3.23), we can substitute in Eq. (3.13) Tikhonov's solution (3.17) and arrive at the following statement.

Theorem 6.6. Let the semi-invariant (3.12) of Eq. (3.4) vanish. Then the series

$$\begin{aligned} u(t,x) =& \mathrm{e}^{-\varrho(t,x)}\Big[F(t) + xF_1(t) + \frac{x^2}{2!}\,F'(t) + \frac{x^3}{3!}\,F_1'(t) + \cdots \\ &+ \frac{x^{2n}}{(2n)!}\,F^{(n)}(t) + \frac{x^{2n+1}}{(2n+1)!}\,F_1^{(n)}(t) + \cdots\Big], \end{aligned} \tag{3.31}$$

where $\varrho(t,x)$ is determined by Eqs. (3.10), solves the equation (3.4).

Example 6.8. Consider again Eq. (3.25). Since here $\varrho(t,x) = -x$, Eq. (3.31) is written

$$u(t,x) = \mathrm{e}^{x}\Big[F(t) + xF_1(t) + \frac{x^2}{2!}\,F'(t) + \frac{x^3}{3!}\,F_1'(t) + \cdots\Big].$$

The solution $u(t,x) = x\mathrm{e}^{x}$ from Example 6.5 corresponds to $F(t) = 0,\ F_1(t) = 1$.

3.6.3 Additional comments

Remark 6.4. Theorems 6.2 and 6.6 can be extended to the parabolic equations

$$\bar{u}_t + \bar{u}_{xx} + \bar{a}(t,x)u_x + \bar{c}(t,x)\bar{u} = 0$$

reducible to the "time reversal heat equation"

$$w_t + w_{xx} = 0$$

by the linear transformation of the dependent variable

$$w = \bar{u}\,\mathrm{e}^{\bar{\varrho}(t,x)},$$

where $\bar{\varrho}$ is obtained by solving the overdetermined system (cf. Eqs. (3.10))

$$\bar{\varrho}_x = \frac{1}{2}\,\bar{a}, \quad \bar{\varrho}_t = \bar{c} - \frac{1}{4}\,\bar{a}^2 - \frac{1}{2}\,\bar{a}_x.$$

The compatibility condition of this system has the form (cf. (3.12))

$$\overline{K} \equiv \bar{a}\bar{a}_x + \bar{a}_{xx} + \bar{a}_t - 2\bar{c}_x = 0.$$

Let us dwell upon the solutions related to Poisson's solution. Since the solution to the "time reversal heat equation" is given by

$$w(t,x) = \frac{1}{2\sqrt{-\pi t}} \int\limits_{-\infty}^{+\infty} f(z)\,\mathrm{e}^{\frac{(x-z)^2}{4t}}\,dz, \quad t < 0,$$

the solution of a reducible equation (i.e. when $\overline{K} = 0$) has the form similar to (3.24):

$$\bar{u}(t,x) = \frac{1}{2\sqrt{-\pi t}}\,\mathrm{e}^{-\bar{\varrho}(t,x)} \int\limits_{-\infty}^{+\infty} f(z)\,\mathrm{e}^{\frac{(x-z)^2}{4t}}\,dz, \quad t < 0.$$

Remark 6.5. If a parabolic equation is given in the general form (2.1),

$$A\,u_{xx} + 2B\,u_{xy} + C\,u_{yy} + a\,u_x + b\,u_y + c\,u = 0, \quad A \neq 0,$$

where $A = A(x,y), \ldots, c = c(x,y)$, we rewrite it in the form (3.1) by introducing the characteristic variable t. Specifically, since $B^2 - AC = 0$, the characteristic equation (2.1) reduces to the linear equation

$$A\,\frac{\partial\varphi}{\partial x} + B\,\frac{\partial\varphi}{\partial y} = 0.$$

Taking any solution $\varphi(x,y)$ of this equation and rewriting the equation in question in the variables x and $t = \varphi(x,y)$, we will arrive at an equation of the form (3.1).

4 Application to financial mathematics

4.1 Transformations of the Black-Scholes equation

Consider the Black-Scholes equation

$$u_t + \frac{1}{2}A^2x^2u_{xx} + Bxu_x - Cu = 0. \quad A, B, C = \text{const.}, \tag{4.1}$$

Upon the change of variables

$$\tau = t_0 - t, \quad y = \frac{\sqrt{2}}{A}\,(\ln|x| - \ln|x_0|), \tag{4.2}$$

it assumes the form of Eq. (3.4) with constant coefficients:

$$u_\tau - u_{yy} + \left(\frac{A}{\sqrt{2}} - \frac{\sqrt{2}}{A}\,B\right)u_y + Cu = 0. \tag{4.3}$$

Therefore, according to Corollary 6.1, the Black-Scholes equation reduces to the heat equation written in the variables (4.2):

$$v_\tau - v_{yy} = 0. \tag{4.4}$$

The corresponding system (3.10) for determining the function $\varrho(\tau, y)$ is written

$$\frac{\partial\varrho}{\partial y} = \frac{B}{\sqrt{2}\,A} - \frac{A}{2\sqrt{2}}\,, \quad \frac{\partial\varrho}{\partial\tau} = C + \frac{A^2}{8} - \frac{B}{2} + \frac{B^2}{2A^2}\,.$$

Integrating this system and ignoring the unessential constant of integration, we have:

$$\varrho = \left(\frac{B}{\sqrt{2}\,A} - \frac{A}{2\sqrt{2}}\right)y + \left(C + \frac{A^2}{8} - \frac{B}{2} + \frac{B^2}{2A^2}\right)\tau.$$

Hence, according to Eq. (3.13), the solution to Eq. (4.3) is given by

$$u(\tau, y) = \mathrm{e}^{\left(\frac{A}{2\sqrt{2}} - \frac{B}{\sqrt{2}\,A}\right)y + \left(\frac{B}{2} - \frac{A^2}{8} - \frac{B^2}{2A^2} - C\right)\tau}\, v(\tau, y), \tag{4.5}$$

where $v(\tau, y)$ is the solution of the heat equation (4.4).

4.2 Poisson's form of the solution to the Black-Scholes equation

According to Theorem 6.4 and Eq. (4.5), the solution to Eq. (4.3) is written

$$u(\tau, y) = \frac{1}{2\sqrt{\pi\tau}}\,\mathrm{e}^{\left(\frac{A}{2\sqrt{2}} - \frac{B}{\sqrt{2}\,A}\right)y + \left(\frac{B}{2} - \frac{A^2}{8} - \frac{B^2}{2A^2} - C\right)\tau} \int_{-\infty}^{+\infty} f(z)\,\mathrm{e}^{-\frac{(y-z)^2}{4\tau}}\,dz, \quad \tau > 0.$$

Substituting the expressions (4.2) for τ and y, we obtain the integral representation of the general solution to the Black-Scholes equation (4.1) in the variables t, x :

$$u(t,x)=\frac{1}{2\sqrt{\pi(t_0-t)}}\;\mathrm{e}^{\left(\frac{1}{2}-\frac{B}{A^2}\right)\ln\left|\frac{x}{x_o}\right|+\left(\frac{B}{2}-\frac{A^2}{8}-\frac{B^2}{2A^2}-C\right)(t_0-t)}$$
$$\times\int\limits_{-\infty}^{+\infty} f(z)\,\mathrm{e}^{-\frac{[\sqrt{2}(\ln|x|-\ln|x_o|)-Az]^2}{4A^2(t_o-t)}}\,dz,\quad t<t_0. \tag{4.6}$$

Black-Scholes [8] gave the solution of the Cauchy problem with a special initial data. The Cauchy problem can also be solved by using the fundamental solution for the Black-Scholes equation obtained in [22] by using the symmetries of the equation. Both ways are not simple. The integral representation (4.6) of the general solution allows one to solve the initial value problem

$$u_t+\frac{1}{2}A^2x^2u_{xx}+Bxu_x-Cu=0\ \ (t<t_0),\quad u\big|_{t=t_0}=u_0(x) \tag{4.7}$$

with an arbitrary initial data $u_0(x)$. Indeed, letting in Eq. (4.6) $t\to t_0$ and using the initial condition we obtain:

$$u_0(x)=\mathrm{e}^{\left(\frac{1}{2}-\frac{B}{A^2}\right)\ln\left|\frac{x}{x_o}\right|}\lim_{t\to t_0}\frac{1}{2\sqrt{\pi(t_0-t)}}\int\limits_{-\infty}^{+\infty} f(z)\,\mathrm{e}^{-\frac{1}{4(t_o-t)}\left[\frac{\sqrt{2}}{A}\ln\left|\frac{x}{x_o}\right|-z\right]^2}dz,$$

whence, according to Eq. (3.29),

$$u_0(x)=f\Big(\frac{1}{A}\sqrt{2}\,\ln\Big|\frac{x}{x_o}\Big|\Big)\,\mathrm{e}^{\left(\frac{1}{2}-\frac{B}{A^2}\right)\ln\left|\frac{x}{x_o}\right|}.$$

Denoting $z=\frac{1}{A}\sqrt{2}\,\ln\left|\frac{x}{x_o}\right|$ we have $x=x_0\mathrm{e}^{\frac{Az}{\sqrt{2}}}$, and the above equation yields

$$f(z)=u_0\Big(x_0\mathrm{e}^{\frac{Az}{\sqrt{2}}}\Big)\,\mathrm{e}^{\left(\frac{B}{\sqrt{2}A}-\frac{A}{2\sqrt{2}}\right)z}.$$

Substituting this expression for f in Eq. (4.6), we obtain the following solution to the problem (4.7):

$$u(t,x)=\frac{1}{2\sqrt{\pi(t_0-t)}}\;\mathrm{e}^{\left(\frac{1}{2}-\frac{B}{A^2}\right)\ln\left|\frac{x}{x_o}\right|+\left(\frac{B}{2}-\frac{A^2}{8}-\frac{B^2}{2A^2}-C\right)(t_0-t)}$$
$$\times\int\limits_{-\infty}^{+\infty} u_0\Big(x_0\mathrm{e}^{\frac{Az}{\sqrt{2}}}\Big)\,\mathrm{e}^{\left(\frac{B}{\sqrt{2}A}-\frac{A}{2\sqrt{2}}\right)z}\,\mathrm{e}^{-\frac{[\sqrt{2}(\ln|x|-\ln|x_o|)-Az]^2}{4A^2(t_o-t)}}\,dz,\quad t<t_0. \tag{4.8}$$

Remark 6.6. Along with the general solution (4.6), group invariant solutions can be useful as well, particularly those given explicitly by elementary functions. Numerous invariant solutions are obtained in [22]. It is not easy to recognize these invariant solutions from Poisson's form (4.6) of the solutions. Compare, e.g. one of the simplest invariant solutions from [22]:

$$u = x\,\mathrm{e}^{(B-C)(t_o-t)}$$

with its integral representation obtained from (4.8) by substituting $u_0(x) = x$:

$$u = \frac{1}{2\sqrt{\pi(t_0-t)}}\ \mathrm{e}^{\left(\frac{1}{2}-\frac{B}{A^2}\right)\ln\left|\frac{x}{x_o}\right| + \left(\frac{B}{2}-\frac{A^2}{8}-\frac{B^2}{2A^2}-C\right)(t_0-t)}$$
$$\times \int_{-\infty}^{+\infty} x_0\,\mathrm{e}^{\left(\frac{B}{\sqrt{2}\,A}+\frac{A}{2\sqrt{2}}\right)z}\,\mathrm{e}^{-\frac{[\sqrt{2}(\ln|x|-\ln|x_o|)-Az]^2}{4A^2(t_o-t)}}\,dz, \quad t < t_0. \tag{4.9}$$

4.3 Tikhonov's form of the solution to the Black-Scholes equation

Substituting in (4.5) the solution $v(\tau, y)$ of the heat equation (4.4) in Tikhonov's form (3.17), we obtain Tikhonov's representation of the solution to Eq. (4.3):

$$u(\tau, y) = \mathrm{e}^{\left(\frac{A}{2\sqrt{2}}-\frac{B}{\sqrt{2}\,A}\right)y + \left(\frac{B}{2}-\frac{A^2}{8}-\frac{B^2}{2A^2}-C\right)\tau}\Big[F(\tau) + yF_1(\tau) + \frac{y^2}{2!}\,F'(\tau)$$
$$+ \frac{y^3}{3!}\,F_1'(\tau) + \cdots + \frac{y^{2n}}{(2n)!}\,F^{(n)}(\tau) + \frac{y^{2n+1}}{(2n+1)!}\,F_1^{(n)}(\tau) + \cdots\Big]. \tag{4.10}$$

Now we replace τ and y by their expressions (4.2) and arrive at the following Tikhonov's form of the solution to the Black-Scholes equation (4.1) (cf. Eq. (4.6)):

$$u(t, x) = \mathrm{e}^{\left(\frac{1}{2}-\frac{B}{A^2}\right)\ln\left|\frac{x}{x_o}\right| + \left(\frac{B}{2}-\frac{A^2}{8}-\frac{B^2}{2A^2}-C\right)(t_0-t)}\Big[F(t_0-t) + F_1(t_0-t)\frac{\sqrt{2}}{A}\,\ln\left|\frac{x}{x_0}\right|$$
$$+ \frac{1}{2!}\,F'(t_0-t)\Big[\frac{\sqrt{2}}{A}\,\ln\left|\frac{x}{x_0}\right|\Big]^2 + \frac{1}{3!}\,F_1'(t_0-t)\Big[\frac{\sqrt{2}}{A}\,\ln\left|\frac{x}{x_0}\right|\Big]^3 + \cdots\Big]. \tag{4.11}$$

Example 6.9. Let us obtain Tikhonov's form of the invariant solution

$$u = x\,\mathrm{e}^{(B-C)(t_o-t)} \tag{4.12}$$

from Remark 6.6. The solution (4.12) is written in the variables τ and y given by (4.2) as follows:

$$u(\tau, y) = x_0\,\mathrm{e}^{\frac{A}{\sqrt{2}}y+(B-C)\tau}. \tag{4.13}$$

We note that if we substitute it in Eq. (4.5),

$$x_0\,\mathrm{e}^{\frac{A}{\sqrt{2}}y+(B_C)\tau} = \mathrm{e}^{\left(\frac{A}{2\sqrt{2}}-\frac{B}{\sqrt{2}\,A}\right)y+\left(\frac{B}{2}-\frac{A^2}{8}-\frac{B^2}{2A^2}-C\right)\tau}\,v(\tau, y),$$

and solve for v we have the following solution to the heat equation (4.4):

$$v(\tau, y) = x_0\,\mathrm{e}^{\alpha y+\alpha^2\tau}, \tag{4.14}$$

where

$$\alpha = \frac{A}{2\sqrt{2}} + \frac{B}{\sqrt{2}A}\cdot$$

Writing

$$x_0\,\mathrm{e}^{\alpha y}\,\mathrm{e}^{\alpha^2\tau} = x_0\,\mathrm{e}^{\alpha^2\tau}\Big(1 + \alpha y + \frac{y^2}{2!}\,\alpha^2 + \frac{y^3}{3!}\,\alpha^3 + \cdots\Big)$$

or

$$x_0\,\mathrm{e}^{\alpha y}\,\mathrm{e}^{\alpha^2\tau} = x_0\,\mathrm{e}^{\alpha^2\tau} + y\alpha\,x_0\,\mathrm{e}^{\alpha^2\tau} + \frac{y^2}{2!}\,\alpha^2\,x_0\,\mathrm{e}^{\alpha^2\tau} + \frac{y^3}{3!}\,\alpha^3\,x_0\,\mathrm{e}^{\alpha^2\tau} + \cdots$$

we see that Tikhonov's series for the solution (4.14):

$$x_0\,\mathrm{e}^{\alpha y}\,\mathrm{e}^{\alpha^2\tau} = F(\tau) + yF_1(\tau) + \frac{y^2}{2!}\,F'(\tau) + \frac{y^3}{3!}\,F_1'(\tau) + \cdots$$

is satisfied with

$$F(\tau) = x_0\,\mathrm{e}^{\alpha^2\tau}, \quad F_1(\tau) = \alpha F(\tau) \equiv \alpha x_0\,\mathrm{e}^{\alpha^2\tau}.$$

Hence, Tikhonov's form (4.11) of the solution (4.12) of the the Black-Scholes equation corresponds to

$$F(t_0 - t) = x_0\,\mathrm{e}^{\alpha^2(t_0-t)}, \quad F_1(t_0 - t) = \alpha x_0\,\mathrm{e}^{\alpha^2(t_0-t)}.$$

4.4 Fundamental and other particular solutions

Eqs. (4.5) and (4.2) provide the transition formula for mapping any exact solution $v(\tau, y)$ of the heat equation (4.4) to an exact solution $u(t, x)$ of the

Black-Scholes equation (4.1). The transition formula can be used either in the form of the system (4.5), (4.2) or in the following explicit form:

$$u(t,x) = \mathrm{e}^{\left(\frac{1}{2}-\frac{B}{A^2}\right)\ln\left|\frac{x}{x_o}\right| + \left(\frac{B}{2}-\frac{A^2}{8}-\frac{B^2}{2A^2}-C\right)(t_0-t)}\, v\Big(t_0-t, \frac{\sqrt{2}}{A}\ln\Big|\frac{x}{x_0}\Big|\Big). \tag{4.15}$$

Let us find the Let us substitute in Eq. (4.15) the fundamental solution the fundamental solution $u = \mathcal{E}(t,x;t_0,x_0)$ of the Cauchy problem for the Black-Scholes equation defined by the equations

$$u_t + \frac{1}{2}A^2x^2u_{xx} + Bxu_x - Cu = 0, \quad t < t_0,$$
$$u\big|_{t=t_0} \equiv u\big|_{t\to t_0} = \delta(x-x_0).$$

We will use the transition formula in the form of the system (4.5), (4.2). If we substitute in (4.5) the function

$$v(\tau,y) = \frac{K}{2\sqrt{\pi\tau}}\,\mathrm{e}^{-\frac{y^2}{4\tau}}, \quad \tau > 0,\ K = \text{const.},$$

which solves the heat equation (4.4) and is proportional to the fundamental solution of the Cauchy problem for the heat equation, we get:

$$u(\tau,y) = \frac{K}{2\sqrt{\pi\tau}}\,\mathrm{e}^{-\frac{y^2}{4\tau}+\left(\frac{A}{2\sqrt{2}}-\frac{B}{\sqrt{2}\,A}\right)y+\left(\frac{B}{2}-\frac{A^2}{8}-\frac{B^2}{2A^2}-C\right)\tau}, \quad \tau > 0. \tag{4.16}$$

Let us replace τ and y by their expressions (4.2), introduce the notation

$$z = \frac{\sqrt{2}}{A}\ln|x|$$

so that $y = z - z_0$, and rewrite (4.16) in the form

$$u = \frac{K}{2\sqrt{\pi(t_0-t)}}\,\mathrm{e}^{-\frac{(z-z_o)^2}{4(t_o-t)}+\left(\frac{A}{2\sqrt{2}}-\frac{B}{\sqrt{2}\,A}\right)(z-z_o)+\left(\frac{B}{2}-\frac{A^2}{8}-\frac{B^2}{2A^2}-C\right)(t_o-t)}, \quad t < t_0.$$

Letting now $t \to t_0$ and proceeding as in Example 6.7 we obtain:

$$u\big|_{t=t_0} = K\mathrm{e}^{\left(\frac{A}{2\sqrt{2}}-\frac{B}{\sqrt{2}\,A}\right)(z-z_o)}\,\delta(z-z_o) = K\delta(z-z_o).$$

Using the formula for the change of variables in the *delta*-function,

$$\delta(x-x_o) = \left|\frac{dz}{dx}\right|_{x=x_0}\delta(z-z_o) = \frac{\sqrt{2}}{A|x_0|}\,\delta(z-z_o)$$

we have:

$$u\big|_{t=t_0} = K\,\frac{A|x_0|}{\sqrt{2}}\,\delta(x-x_o).$$

Therefore, we take

$$K = \frac{\sqrt{2}}{A|x_0|}\,,$$

substitute the expression for z and obtain the following fundamental solution of the Cauchy problem for the Black-Scholes equation:

$$\mathcal{E} = \frac{1}{A|x_0|\sqrt{2\pi(t_0-t)}}\,\mathrm{e}^{-\frac{(\ln|x|-\ln|x_o|)^2}{2A^2(t_0-t)}+\left(\frac{1}{2}-\frac{B}{A^2}\right)\ln\left|\frac{x}{x_o}\right|+\left(\frac{B}{2}-\frac{A^2}{8}-\frac{B^2}{2A^2}-C\right)(t_0-t)}, \tag{4.17}$$

where $t < t_0$. It was obtained in [22] by means of the invariance principle.

The transition formula (4.15) can be used for obtaining numerous particular solutions to the Black-Scholes equation in a closed form, in particular, those given by elementary functions. For example, one can take for v any invariant solution of the heat equation. In this way, one can obtain all invariant solutions of the Black-Scholes equation by enumerating independent invariant solutions of the heat equation provided by the optimal system of one-dimensional subalgebras of the symmetry algebra of the heat equation. Another way to find particular solutions of the Black-Scholes equation is to substitute in Eq. (4.11) any polynomials F and F_1.

Example 6.10. The infinitesimal symmetry

$$X = \frac{\partial}{\partial\tau} - \alpha^2 v\frac{\partial}{\partial v}\,, \quad \alpha = \text{const.},$$

of the heat equation $v_\tau - v_{yy} = 0$ provides the following invariant solution:

$$v = \big[C_1\cos(\alpha y) + C_2\sin(\alpha y)\big]\mathrm{e}^{-\alpha^2\tau}, \quad C_1, C_2 = \text{const.}$$

Substituting it in (4.15) we obtain the following solution to the Black-Scholes equation:

$$u = \left[C_1\cos\left(\beta\ln\left|\frac{x}{x_0}\right|\right)+C_2\sin\left(\beta\ln\left|\frac{x}{x_0}\right|\right)\right]\mathrm{e}^{\left(\frac{1}{2}-\frac{B}{A^2}\right)\ln\left|\frac{x}{x_o}\right|+\left(\frac{B}{2}-\frac{A^2}{8}-\frac{B^2}{2A^2}-C-\alpha^2\right)(t_0-t)},$$

where $\beta = \frac{\alpha\sqrt{2}}{A}$.

4.5 Non-linearization of the Black-Scholes model

Recall that the heat equation (4.4) is connected with the Burgers equation

$$w_\tau - ww_y - w_{yy} = 0 \tag{4.18}$$

by the following differential substitution known as the Hopf-Cole transformation:

$$w = 2\frac{v_y}{v} \equiv \frac{\partial \ln v^2}{\partial y}\,\cdot \tag{4.19}$$

Eq. (4.18) has specific properties due to its nonlinearity and is used in turbulence theories, non-linear acoustics, etc. It seems reasonable to employ the remarkable connections of the heat equation with the Black-Scholes and the Burgers equations in order to describe nonlinear effects in finance ("financial turbulence"). Noting that the transformations (4.2) and (4.5) formally bring in the heat equation a "financial content", I rewrite the Hopf-Cole transformation (4.19) in the variables t, x, u given by (4.2), (4.5) and the Burgers equation (4.18) in the variables t, x given by (4.2) and obtain:

$$w = \sqrt{2}\Big(\frac{B}{A} - \frac{A}{2} + Ax\frac{u_x}{u}\Big), \tag{4.20}$$

$$w_t + \frac{A}{\sqrt{2}}\left(w + \frac{A}{\sqrt{2}}\right) xw_x + \frac{A^2}{2}\,x^2 w_{xx} = 0. \tag{4.21}$$

One can verify that the linear Black-Scholes equation is connected with the nonlinear equation (4.21) by the transformation (4.20).

4.6 Symmetries of the basic equations

For the convenience of the reader who might be interested in investigating invariant solutions of the Black-Scholes equation and/or of the nonlinear equation (4.21), I will list first of all the well-known symmetries of the heat equation (4.4) and of the Burgers equation (4.18). Then I will recall the symmetries of the Black-Scholes equation computed in [22] and will give the symmetries of the nonlinear equation (4.21). All these symmetries are mutually connected by the transformations (4.2), (4.5), (4.19) and (4.20). For the linear equations (4.4) and (4.1) I will omit the trivial infinite part of their symmetry algebras appearing due to the superposition principle. For example, for the heat equation this trivial part comprises the operator

$$X_* = v_*(\tau, y)\frac{\partial}{\partial v}\,,$$

where $v_*(\tau, y)$ is any particular solution of Eq. (4.4).

4.6.1 Symmetries of the heat and Burgers equations

The symmetries of the heat equation (4.4) are:

$$X_1 = \frac{\partial}{\partial \tau}\,, \quad X_2 = \frac{\partial}{\partial y}\,, \quad X_3 = 2\tau\frac{\partial}{\partial \tau} + y\frac{\partial}{\partial y}\,, \quad X_4 = v\frac{\partial}{\partial v}\,, \tag{4.22}$$

$$X_5 = 2\tau\frac{\partial}{\partial y} - yv\frac{\partial}{\partial v}\,, \quad X_6 = \tau^2\frac{\partial}{\partial \tau} + \tau y\frac{\partial}{\partial y} - \frac{1}{4}(2\tau + y^2)\,v\frac{\partial}{\partial v}\,\cdot$$

The symmetries of the Burgers equation (4.18) are:

$$X_1 = \frac{\partial}{\partial \tau}\,, \quad X_2 = \frac{\partial}{\partial y}\,, \quad X_3 = 2\tau\frac{\partial}{\partial \tau} + y\frac{\partial}{\partial y} - w\frac{\partial}{\partial w}\,, \tag{4.23}$$

$$X_4 = \tau\frac{\partial}{\partial y} - \frac{\partial}{\partial w}\,, \quad X_5 = \tau^2\frac{\partial}{\partial \tau} + \tau y\frac{\partial}{\partial y} - (y + \tau w)\frac{\partial}{\partial w}\,\cdot$$

To obtain the symmetries (4.23) of the Burgers equation there is no need to solve the determining equations. They can be obtained merely by rewriting the symmetries (4.22) of the heat equation in terms of the variable w given by Eq. (4.19). I will illustrate the procedure by the operators X_5 and X_6 from (4.22).

Let us begin with X_5. We extend it to v_y by the usual prolongation formula,

$$X_5 = 2\tau\frac{\partial}{\partial y} - yv\frac{\partial}{\partial v} - (v + yv_y)\frac{\partial}{\partial v_y}\,,$$

and act on the dependent variable w of the Burgers equation. Using the definition (4.19) of w we have:

$$X_5(w) = -\frac{2}{v}\,\big(v + yv_y\big) + 2(yv)\frac{v_y}{v^2} = -2.$$

Hence, X_5 is written in terms of the variables τ, y and w of the Burgers equations as follows:

$$X_5 = 2\Big[\tau\frac{\partial}{\partial y} - \frac{\partial}{\partial w}\Big].$$

Up to the unessential coefficient 2, this is the operator X_4 from (4.23).

Proceeding likewise with the operator X_6 we obtain:

$$X_6 = \tau^2\frac{\partial}{\partial \tau} + \tau y\frac{\partial}{\partial y} - \frac{1}{4}(2\tau + y^2)v\frac{\partial}{\partial v} - \Big[\frac{1}{4}(2\tau + y^2)v_y + \frac{1}{2}yv + \tau v_y\Big]\frac{\partial}{\partial v_y}\,,$$

$$X_6(w) = -y - 2\tau\frac{v_y}{v} = -(y + \tau w),$$

and hence we get

$$X_6 = \tau^2 \frac{\partial}{\partial \tau} + \tau y \frac{\partial}{\partial y} - (y + \tau w) \frac{\partial}{\partial w}\,,$$

i.e. the operator X_5 from (4.23).

If we will perform the similar procedure with the operator X_4 from (4.22), we will see that its action on the variables of the Burgers equation is identically zero. Indeed, its prolongation to v_y is written

$$X_4 = v \frac{\partial}{\partial v} + v_y \frac{\partial}{\partial v_y}\,,$$

and hence $X_4(w) = 0$. Therefore X_4 does not provide any symmetry for the Burgers equation, and hence the Burgers equation possesses only five Lie point symmetries.

4.6.2 Symmetries of the Black-Scholes equation

The Black-Scholes equation has the following symmetries obtained in [22] by solving the determining equations. They can also be obtained by subjecting the symmetries (4.22) of the heat equation to the transformations (4.2), (4.5).

$$\begin{aligned}
&X_1 = \frac{\partial}{\partial t}\,, \quad X_2 = x \frac{\partial}{\partial x}\,, \quad X_3 = 2t \frac{\partial}{\partial t} + \big(\ln x + Pt\big) x \frac{\partial}{\partial x} + 2Ctu \frac{\partial}{\partial u}\,,\\
&X_4 = u \frac{\partial}{\partial u}\,, \quad X_5 = A^2 tx \frac{\partial}{\partial x} + \big(\ln x - Pt\big) u \frac{\partial}{\partial u}\,, \qquad (4.24)\\
&X_6 = 2A^2t^2 \frac{\partial}{\partial t} + 2A^2 tx \ln x \frac{\partial}{\partial x} + \Big[\big(\ln x - Pt\big)^2 + 2A^2Ct^2 - A^2 t\Big] u \frac{\partial}{\partial u}\,,
\end{aligned}$$

where P is the constant defined by

$$P = B - \frac{1}{2} A^2.$$

4.6.3 Symmetries of the nonlinear equation (4.21)

Let us obtain the symmetries of the nonlinear equation (4.21) by subjecting the symmetries (4.23) of the Burgers equation to the transformation (4.2) written in the form

$$t = t_0 - \tau, \quad x = x_0 \mathrm{e}^{\frac{Ay}{\sqrt{2}}}. \qquad (4.25)$$

It is manifest that the transformation (4.25) maps the first operator (4.23) into

$$X_1 = \frac{\partial}{\partial t}\,\cdot$$

For the second operator (4.23) we have

$$X_2(x) = \frac{A}{\sqrt{2}}\, x_0 \mathrm{e}^{\frac{Ay}{\sqrt{2}}} = \frac{A}{\sqrt{2}}\, x.$$

Hence, ignoring the unessential constant factor we obtain

$$X_2 = x\frac{\partial}{\partial x}\,\cdot$$

For the third operator (4.23) we have

$$X_3(t) = -2\tau = 2(t - t_0), \quad X_3(x) = \frac{A}{\sqrt{2}}\, yx = x(\ln|x| - \ln|x_0|).$$

Therefore the operator X_3 from (4.23) becomes

$$2(t - t_0)\frac{\partial}{\partial t} + x(\ln|x| - \ln|x_0|)\frac{\partial}{\partial x} - w\frac{\partial}{\partial w}$$

$$= 2t\frac{\partial}{\partial t} + x\ln|x|\frac{\partial}{\partial x} - w\frac{\partial}{\partial w} - 2t_0 X_1 - \ln|x_0| X_2,$$

where X_1 and X_2 are given above. Taking it modulo X_1, X_2, we obtain the following operator admitted by Eq. (4.21):

$$X_3 = 2t\frac{\partial}{\partial t} + x\ln|x|\frac{\partial}{\partial x} - w\frac{\partial}{\partial w}\,\cdot$$

Proceeding likewise with remaining two operators from (4.23) we arrive at the following symmetries of Eq. (4.21):

$$\begin{aligned} &X_1 = \frac{\partial}{\partial t}\,, \quad X_2 = x\frac{\partial}{\partial x}\,, \quad X_3 = 2t\frac{\partial}{\partial t} + x\ln|x|\frac{\partial}{\partial x} - w\frac{\partial}{\partial w}\,,\\ &X_4 = Atx\frac{\partial}{\partial x} + \sqrt{2}\,\frac{\partial}{\partial w}\,, \\ &X_5 = t^2\frac{\partial}{\partial t} + tx\ln|x|\frac{\partial}{\partial x} + \left(\frac{\sqrt{2}}{A}\,\ln|x| - tw\right)\frac{\partial}{\partial w}\,\cdot \end{aligned} \tag{4.26}$$

4.7 Optimal system of one-dimensional subalgebras for Equation (4.21)

The structure of the Lie algebra L_5 spanned by the operators (4.26) is described by the following commutator table.

	X_1	X_2	X_3	X_4	X_5
X_1	0	0	$2X_1$	AX_2	X_3
X_2	0	0	X_2	0	$\frac{1}{A}X_4$
X_3	$-2X_1$	$-X_2$	0	X_4	$2X_5$
X_4	$-AX_2$	0	$-X_4$	0	0
X_5	$-X_3$	$-\frac{1}{A}X_4$	$-2X_5$	0	0

(4.27)

The Lie algebra L_5 spanned the symmetries (4.26) provides a possibility to find invariant solutions of Eq. (4.21) based on any one-dimensional subalgebra of the algebra L_5, i.e. on any operator $X \in L_5$. However, there are infinite number of one-dimensional subalgebras of L_5 since an arbitrary operator from L_5 is written

$$X = l^1 X_1 + \ldots + l^5 X_5, \tag{4.28}$$

and hence depends on five arbitrary constants $l^1, \ldots, l^5$. In order to make this problem manageable, L.V. Ovsyannikov [91] (see also [92]) introduced the concept of *optimal systems of subalgebras** by noting that if two subalgebras are *similar*, i.e. connected with each other by a transformation of the symmetry group, then their corresponding invariant solutions are connected with each other by the same transformation. Consequently, it is sufficient to deal with an optimal system of invariant solutions obtained, in our case, as follows. We put into one class all similar operators $X \in L_5$ and select a representative of each class. The set of the representatives of all these classes is an *optimal system of one-dimensional subalgebras.*

Now all invariant solutions can in principle be obtained by constructing the invariant solution for each member of the optimal system of subalgebras. The set of invariant solutions obtained in this way is an *optimal system of invariant solutions.* It is worth noting that the form of these invariant

*He considered subalgebras of any dimension. We need here only one-dimensional subalgebras.

solutions depends on the choice of representatives. If one constructs an optimal system of invariant solutions and subjects these solutions to all transformations of the group admitted by the equation in question, one obtains *all invariant solutions* of this equation.

Let us construct an optimal system of one-dimensional subalgebras of the Lie algebra L_5 following the simple method used by Ovsyannikov [91].

The transformations of the symmetry group with the Lie algebra L_5 provide the 5-parameter group of linear transformations of the operators $X \in L_5$ (see examples in Section 4.8.12) or, equivalently, linear transformations of the vector

$$l = (l^1, \ldots, l^5), \tag{4.29}$$

where $l^1, \ldots, l^5$ are taken from (4.28). To find these linear transformations, we use their generators (see, e.g. [36], Section 1.4)

$$E_\mu = c^\lambda_{\mu\nu} l^\nu \frac{\partial}{\partial l^\lambda}\,, \quad \mu = 1, \ldots, 5, \tag{4.30}$$

where $c^\lambda_{\mu\nu}$ are the structure constants of the Lie algebra L_5 defined by

$$[X_\mu, X_\nu] = c^\lambda_{\mu\nu} X_\lambda.$$

Let us find, e.g. the operator E_1. According to (4.30), it is written

$$E_1 = c^\lambda_{1\nu} l^\nu \frac{\partial}{\partial l^\lambda}\,,$$

where $c^\lambda_{1\nu}$ are defined by the commutators $[X_1, X_\nu] = c^\lambda_{\mu\nu} X_\lambda$, i.e. by the first raw in Table (4.27). Namely, the non-vanishing $c^\lambda_{\mu\nu}$ are

$$c^1_{13} = 2, \quad c^2_{14} = A, \quad c^2_{15} = 1.$$

Therefore we have:

$$E_1 = 2l^3 \frac{\partial}{\partial l^1} + A l^4 \frac{\partial}{\partial l^2} + l^5 \frac{\partial}{\partial l^3}\,\cdot$$

Substituting in (4.30) all structure constants given by Table (4.27) we obtain:

$$E_1 = 2l^3 \frac{\partial}{\partial l^1} + A l^4 \frac{\partial}{\partial l^2} + l^5 \frac{\partial}{\partial l^3}\,, \qquad E_2 = l^3 \frac{\partial}{\partial l^2} + \frac{1}{A}\, l^5 \frac{\partial}{\partial l^4}\,,$$

$$E_3 = -2l^1 \frac{\partial}{\partial l^1} - l^2 \frac{\partial}{\partial l^2} + l^4 \frac{\partial}{\partial l^4} + 2l^5 \frac{\partial}{\partial l^5}\,, \quad E_4 = -A l^1 \frac{\partial}{\partial l^2} - l^3 \frac{\partial}{\partial l^4}\,,$$

$$E_5 = -l^1 \frac{\partial}{\partial l^3} - \frac{1}{A}\, l^2 \frac{\partial}{\partial l^4} - 2l^3 \frac{\partial}{\partial l^5}\,\cdot \tag{4.31}$$

Let us find the transformations provided by the generators (4.31). For the generator E_1, the Lie equations with the parameter a_1 are written

$$\frac{d\tilde{l}^1}{da_1} = 2\tilde{l}^3, \quad \frac{d\tilde{l}^2}{da_1} = A\tilde{l}^4, \quad \frac{d\tilde{l}^3}{da_1} = \tilde{l}^5, \quad \frac{d\tilde{l}^4}{da_1} = 0, \quad \frac{d\tilde{l}^5}{da_1} = 0.$$

Integrating these equations and using the initial condition $\tilde{l}\,\big|_{a_1=0} = l$, we obtain:

$$E_1: \quad \tilde{l}^{\,1} = l^1 + 2a_1 l^3 + a_1^2 l^5, \quad \tilde{l}^{\,2} = l^2 + a_1 A l^4,$$
$$\tilde{l}^{\,3} = l^3 + a_1 l^5, \qquad \tilde{l}^{\,4} = l^4, \qquad \tilde{l}^{\,5} = l^5. \tag{4.32}$$

Taking the other operators (4.31) we obtain the following transformations:

$$E_2: \quad \tilde{l}^{\,1} = l^1, \ \ \tilde{l}^{\,2} = l^2 + a_2 l^3, \ \ \tilde{l}^{\,3} = l^3, \ \ \tilde{l}^{\,4} = l^4 + \frac{a_2}{A}\, l^5, \quad \tilde{l}^{\,5} = l^5; \tag{4.33}$$

$$E_3: \quad \tilde{l}^{\,1} = a_3^{-2}\, l^1, \quad \tilde{l}^{\,2} = a_3^{-1}\, l^2, \quad \tilde{l}^{\,3} = l^3, \quad \tilde{l}^{\,4} = a_3 l^4, \quad \tilde{l}^{\,5} = a_3^2 l^5, \tag{4.34}$$

where $a_3 > 0$ since the integration of the Lie equations yields, e.g. $l^4 = l^4 \mathrm{e}^{\tilde{a}_3} = a_3 l^4$. We have further

$$E_4: \quad \tilde{l}^{\,1} = l^1, \quad \tilde{l}^{\,2} = l^2 - a_4 A l^1, \quad \tilde{l}^{\,3} = l^3,$$
$$\tilde{l}^{\,4} = l^4 - a_4 l^3, \quad \tilde{l}^{\,5} = l^5; \tag{4.35}$$

$$E_5: \quad \tilde{l}^{\,1} = l^1, \qquad \tilde{l}^{\,2} = l^2, \qquad \tilde{l}^{\,3} = l^3 - a_5 l^1,$$
$$\tilde{l}^{\,4} = l^4 - \frac{a_5}{A}\, l^2, \quad \tilde{l}^{\,5} = l^5 - 2a_5 l^3 + a_5^2 l^1. \tag{4.36}$$

Note that the transformations (4.32)-(4.36) map the vector $X \in L_5$ given by (4.28) to the vector $\widetilde{X} \in L_5$ given by the following formula:

$$\widetilde{X} = \tilde{l}^{\,1} X_1 + \ldots + \tilde{l}^{\,5} X_5. \tag{4.37}$$

Now we can prove the following result on an optimal system of one-dimensional subalgebras of the five-dimensional Lie algebra of symmetries of Eq. (4.21).

Theorem 6.7. The following operators provide an optimal system of one-dimensional subalgebras of the Lie algebra L_5 with the basis (4.26):

$$X_1, \quad X_2, \quad X_3, \quad X_4, \quad X_1 + X_4, \quad X_1 - X_4,$$
$$X_5, \qquad X_1 + X_5, \qquad X_2 + X_5, \qquad X_2 - X_5. \tag{4.38}$$

Proof. We first clarify if the transformations (4.32)-(4.36) have invariants $J(l^1,\ldots,l^5)$. The reckoning shows that the 5×5 matrix $\|c^\lambda_{\mu\nu}l^\nu\|$ of the coefficients of the operators (4.31) has the rank four. It means that the transformations (4.32)-(4.36) have precisely one functionally independent invariant. The integration of the equations

$$E_\mu(J)=0,\quad \mu=1,\ldots,5,$$

shows that the invariant is

$$J=(l^3)^2-l^1l^5. \tag{4.39}$$

Knowledge of the invariant (4.39) simplifies further calculations.

Since X_5 is the most complicated operator among the symmetries (4.26), we will try to exclude it from the operators of the optimal system when it is possible. In other words, we have to annul $\bar{l}_5$ if possible. It is manifest from Eqs. (4.32)-(4.36) that we can annul $\bar{l}_5$ only by the transformation (4.36) provided that l^1 and l^3 do not vanish simultaneously.

The last equation in (4.36) shows that if $l^1\neq 0$, we get $\bar{l}_5=0$ by solving the quadratic equation $l^5-2a_5l^3+a_5^2l^1=0$ for a_5, i.e. by taking

$$a_5=\frac{l^3\pm\sqrt{J}}{l^1}\,, \tag{4.40}$$

where J is the invariant (4.39). We can use Eq. (4.40) only if $J\geq 0$.

Now we begin the construction of the optimal system. The method requires a simplification of the general vector (4.29) by means of the transformations (4.32)-(4.36). As a result, we will find the simplest representatives of each class of similar vectors (4.29). Substituting these representatives in (4.28), we will obtain the optimal system of one-dimensional subalgebras of L_5. We will divide the construction to several cases.

4.7.1 The case $l^1=0$

I will divide this case into the following two subcases.

1°. $l^3\neq 0$. In other words, we consider the vectors (4.29) of the form

$$(0,l^2,l^3,l^4,l^5),\quad l^3\neq 0.$$

First we take $a_5=l^5/(2l^3)$ in (4.36) and reduce the above vector to

$$(0,l^2,l^3,l^4,0).$$

Then we use the transformation (4.35) with $a_4 = l^4/l^3$ and map the latter vector into the vector

$$(0, l^2, l^3, 0, 0).$$

Now we subject this vector to the transformation E_2 given by (4.33) with $a_2 = -l^2/l^3$ and arrive at the vector

$$(0, 0, l^3, 0, 0).$$

Since the operator X is defined up to a constant factor and $l^3 \neq 0$, we divide the above vector by l^3 and transform it to the form

$$(0, 0, 1, 0, 0).$$

Substituting it in (4.28), we obtain the operator

$$X_3. \tag{4.41}$$

2°. $l^3 = 0$. Thus, we consider the vectors (4.29) of the form

$$(0, l^2, 0, l^4, l^5).$$

2°(1). If $l^2 \neq 0$, we can assume $l^2 = 1$ (see above), use the transformation (4.36) with $a_5 = Al^4$ and get the vector

$$(0, 1, 0, 0, l^5).$$

If $l^5 \neq 0$ we can make $l^5 = \pm 1$ by the transformation (4.34). Thus, taking into account the possibility $l^5 = 0$, we obtain the following representatives for the optimal system:

$$X_2, \quad X_2 + X_5, \quad X_2 - X_5. \tag{4.42}$$

2°(2). Let $l^2 = 0$. If $l^5 \neq 0$ we can set $l^5 = 1$. Now we apply the transformation (4.33) with $a_2 = -Al^4$ and obtain the vector $(0, 0, 0, 0, 1)$. If $l^5 = 0$ we get the vector $(0, 0, 0, 1, 0)$. Thus, the case $l^2 = 0$ provides the operators

$$X_4, \quad X_5. \tag{4.43}$$

4.7.2 The case $l^1 \neq 0, \ J > 0$

Now we can define a_5 by Eq. (4.40) and annul $\bar{l}^5$ by the transformation (4.36). Thus, we will deal with the vector

$$(l^1, l^2, l^3, l^4, 0), \quad l^1 \neq 0.$$

Since J is invariant under the transformations (4.32)-(4.36), the condition $J > 0$ yields that in the above vector we have $l^3 \neq 0$. Therefore we can use the transformation (4.35) with $a_4 = l^4/l^3$ and get $\bar{l}^4 = 0$. Then we apply the transformation (4.32) with $a_1 = -l^1/(2l^3)$ and obtain $\bar{l}^1 = 0$, thus arriving at the vector $(0, l^2, l^3, 0, 0)$, and hence at the previous operator (4.41). Hence, this case contributes no additional subgroups to the optimal system.

4.7.3 The case $l^1 \neq 0, \ J = 0$

In this case Eq. (4.40) reduces to $a_5 = l^3/l^1$.

If $l^3 \neq 0$, we use the transformation (4.36) with $a_5 = l^3/l^1$ and obtain $\bar{l}^5 = 0$. Due to the invariance of J we conclude that the equation $J = 0$ yields $(\bar{l}^3)^2 - \bar{l}^1\bar{l}^5 = 0$. Since $\bar{l}^5 = 0$, it follows that $\bar{l}^3 = 0$. Thus we can deal with the vectors of the form

$$(l^1, l^2, 0, l^4, 0), \quad l^1 \neq 0. \tag{4.44}$$

Furthermore, if $l^3 = 0$, we have $J = -l^1 l^5$, and the equation $J = 0$ yields $l^5 = 0$ since $l^1 \neq 0$. Therefore we again have the vectors of the form (4.44) where we can assume $l^1 = 1$. Subjecting the vector (4.44) with $l^1 = 1$ to the transformation (4.35) with $a_4 = l^2/A$ we obtain $\bar{l}^2 = 0$, and hence map the vector (4.44) to the form

$$(1, 0, 0, l^4, 0).$$

If $l^4 \neq 0$, we use the transformation (4.34) with an appropriately chosen a_3 and obtain $l^4 = \pm 1$. taking into account the possibility $l^4 = 0$, we see that this case contributes the following operators:

$$X_1, \quad X_1 + X_4, \quad X_1 - X_4. \tag{4.45}$$

4.7.4 The case $l^1 \neq 0, \ J < 0$

It is obvious from the condition $J = (l^3)^2 - l^1 l^5 < 0$ that $l^5 \neq 0$. Therefore we successively apply the transformations (4.36), (4.35) and (4.33) with $a_5 = l^3/l^1$, $a_4 = l^2/(Al^1)$ and $a_2 = -Al^4/l^5$, respectively and obtain $\bar{l}^3 = \bar{l}^2 = \bar{l}^4 = 0$. The components l^1 and l^5 of the resulting vector

$$(l^1, 0, 0, 0, l^5)$$

have the common sign since the condition $J < 0$ yields $l^1 l^5 > 0$. Therefore using the transformation (4.34) with an appropriate value of the parameter

a_3 and invoking that we can multiply the vector l by any constant, we obtain $l^1 = l^5 = 1$, i.e. the operator

$$X_1 + X_5. \tag{4.46}$$

Finally, collecting the operators (4.41), (4.42), (4.43), (4.45), (4.46), we arrive at the optimal system (4.38) and complete the proof of the theorem.

4.8 Invariant solutions of Equation (4.21)

In order to construct an optimal system of invariant solutions, we have to find the invariant solution for each operator of the optimal system (4.38). For the sake of simplicity, we will make the calculations for positive values of the variables t and x.

4.8.1 Invariant solution for the operator X_1

Two functionally independent invariants for the operator X_1 are x and w. Consequently, the invariant solution is the stationary solution, $w = w(x)$. For this solution, Eq. (4.21) yields the following second-order ordinary differential equation:

$$\left(w + \frac{A}{\sqrt{2}}\right) xw' + \frac{A}{\sqrt{2}}\, x^2 w'' = 0. \tag{4.47}$$

By setting

$$\psi = \frac{\sqrt{2}}{A}\left(w + \frac{A}{\sqrt{2}}\right)$$

or

$$w = \frac{A}{\sqrt{2}}(\psi - 1),$$

and considering ψ as a function $\psi = \psi(z)$ of the new independent variable

$$z = \ln x,$$

we rewrite Eq. (4.47) in the form

$$\psi'' - \psi' + \psi\psi' = 0.$$

Integrating it once, we obtain

$$\psi' - \psi + \frac{1}{2}\,\psi^2 = \frac{1}{2}\,K_1. \tag{4.48}$$

The general solution to Eq. (4.48) is given by quadrature:

$$\int \frac{2d\psi}{\psi^2 - 2\psi - K_1} = -(z + K_2), \tag{4.49}$$

and can be written in terms of elementary fuinctions. Namely, consider the equation

$$\psi^2 - 2\psi - K_1 = 0.$$

According to the solutuion formula $\psi = 1 \pm \sqrt{1 + K_1}$ we deal with three cases:

$$\text{(i) } K_1 = -1, \quad \text{(ii) } 1 + K_1 = \alpha^2 > 0, \quad \text{(iii) } 1 + K_1 = -\alpha^2 < 0.$$

In the first case we have $\psi^2 - 2\psi - K_1 = (\psi - 1)^2$ and the integral in (4.49) is

$$\int \frac{2d\psi}{\psi^2 - 2\psi - K_1} = \int \frac{2d\psi}{(\psi - 1)^2} = -\frac{2}{\psi - 1}\,\cdot$$

Therefore Eq. (4.49) yields

$$\psi - 1 = \frac{2}{z + K_2}\,\cdot$$

Thus, the first case leads to the following solution of Eq. (4.48):

$$\text{(i)} \qquad w = \frac{A\sqrt{2}}{K_2 + \ln x}\,\cdot$$

In the second case we have $\psi^2 - 2\psi - K_1 = (\psi - 1 - \alpha)(\psi - 1 + \alpha)$, and hence

$$\int \frac{2d\psi}{\psi^2 - 2\psi - K_1} = \frac{1}{\alpha}\Big[\int \frac{d\psi}{\psi - 1 - \alpha} - \int \frac{d\psi}{\psi - 1 + \alpha}\Big] = -\frac{1}{\alpha}\,\ln \frac{\psi - 1 + \alpha}{\psi - 1 - \alpha}\,\cdot$$

Therefore Eq. (4.49) yields

$$\psi - 1 = \alpha\,\frac{\beta e^{\alpha z} + 1}{\beta e^{\alpha z} - 1}\,, \quad \beta = \text{const.}$$

Thus, the second case leads to the following solution of Eq. (4.48):

$$\text{(ii)} \qquad w = \frac{A\alpha}{\sqrt{2}}\,\frac{\beta x^\alpha + 1}{\beta x^\alpha - 1}\,, \quad \alpha, \beta = \text{const.}$$

In the third case the integral in (4.49) is written

$$\int \frac{2d\psi}{\psi^2 - 2\psi + 1 + \alpha^2} = \frac{2}{\alpha}\,\arctan\Big(\frac{\psi - 1}{\alpha}\Big).$$

Therefore Eq. (4.49) yields

$$\psi - 1 = \alpha \tan\Big(-\frac{\alpha}{2}\,(z + K_2)\Big).$$

Thus, the first case leads to the following solution of Eq. (4.48):

$$\text{(iii)} \qquad w = \frac{A\alpha}{\sqrt{2}}\,\tan\Big(\beta - \frac{\alpha}{2}\,\ln x\Big).$$

Summing up, we conclude that the operator X_1 provides the invariant solution given by the following formulae:

$$\begin{aligned} &\text{(i)} \qquad w = \frac{A\sqrt{2}}{K + \ln x}\,;\\ &\text{(ii)} \qquad w = \frac{A\alpha}{\sqrt{2}}\,\frac{\beta x^{\alpha} + 1}{\beta x^{\alpha} - 1}\,;\\ &\text{(iii)} \qquad w = \frac{A\alpha}{\sqrt{2}}\,\tan\Big(\beta - \frac{\alpha}{2}\,\ln x\Big). \end{aligned} \tag{4.50}$$

4.8.2 Invariant solution for the operator X_2

The operator X_2 from the optimal system (4.38) yields $w = w(t)$ and provides the trivial invariant solution

$$w = K, \quad K = \text{const.} \tag{4.51}$$

It can be obtained from the solution (4.50)(ii) by letting $\beta = 0$ and α be arbitrary.

4.8.3 Invariant solution for the operator X_3

The operator

$$X_3 = 2t\frac{\partial}{\partial t} + x\ln x\frac{\partial}{\partial x} - w\frac{\partial}{\partial w}$$

has the invariants

$$\lambda = \frac{\ln x}{\sqrt{t}}\,, \quad \varphi = \sqrt{t}\,w.$$

Therefore the candidates for the invariant solutions are written

$$w = \frac{\varphi(\lambda)}{\sqrt{t}}\,, \quad \lambda = \frac{\ln x}{\sqrt{t}}\,\cdot \tag{4.52}$$

The reckoning shows that

$$w_t = -\frac{1}{2t\sqrt{2}}\Big(\varphi + \lambda\varphi'\Big), \quad w_x = \frac{\varphi'}{tx}, \quad w_{xx} = \frac{1}{x^2}\left(\frac{\varphi''}{t\sqrt{2}} - \frac{\varphi'}{t}\right).$$

Substituting (4.52) and the above expressions for the derivatives in Eq. (4.21) we obtain:

$$\frac{A^2}{2t\sqrt{t}}\left[\varphi'' + \frac{\sqrt{2}}{A}\,\varphi\varphi' - \frac{1}{A^2}(\varphi + \lambda\varphi')\right] = 0,$$

whence

$$\varphi'' + \frac{\sqrt{2}}{A}\,\varphi\varphi' - \frac{1}{A^2}\big(\varphi + \lambda\varphi'\big) = \left(\varphi' + \frac{1}{A\sqrt{2}}\,\varphi^2 - \frac{1}{A^2}\,\lambda\varphi\right)' = 0.$$

Integrating the above equation once, we obtain the Riccati equation

$$\varphi' + \frac{1}{A\sqrt{2}}\,\varphi^2 - \frac{1}{A^2}\,\lambda\varphi = K, \quad K = \text{const.} \tag{4.53}$$

Thus, the invariant solution for the operator X_3 has the form (4.52), where the function $\varphi(\lambda)$ is defined by Eq. (4.53).

4.8.4 Invariant solution for the operator X_4

The invariants for the operator X_4 are t and

$$\varphi = tw - \frac{\sqrt{2}}{A}\,\ln x.$$

Therefore the candidates for the invariant solutions have the form

$$w = \frac{\sqrt{2}}{A}\,\frac{\ln x}{t} + \varphi(t).$$

One can easily verify that Eq. (4.21) yields

$$\varphi' + \frac{\varphi}{t} = 0,$$

and hence $\varphi(t) = K/t$. Thus, X_4 provides the following invariant solution

$$w = \frac{1}{t}\left(\frac{\sqrt{2}}{A}\,\ln x + K\right), \quad K = \text{const.} \tag{4.54}$$

4.8.5 Invariant solution for the operator $X_1 + X_4$

The operator

$$X_1 + X_4 = \frac{\partial}{\partial t} + Atx\frac{\partial}{\partial x} + \sqrt{2}\frac{\partial}{\partial w}$$

has the invariants

$$\lambda = \ln x - \frac{A}{2}\,t^2, \quad \varphi = w - \sqrt{2}\,t.$$

The candidates for the invariant solutions are obtained by letting $\varphi = \varphi(\lambda)$. Hence

$$w = \sqrt{2}\,t + \varphi(\lambda), \quad \lambda = \ln x - \frac{A}{2}\,t^2. \tag{4.55}$$

Calculating the derivatives:

$$w_t = \sqrt{2} - At\varphi', \quad w_x = \frac{1}{x}\,\varphi', \quad w_{xx} = \frac{1}{x^2}\left(\varphi'' - \varphi'\right)$$

and substituting in Eq. (4.21) we obtain for $\varphi(\lambda)$ the following ordinary differential equation:

$$\varphi'' + \frac{\sqrt{2}}{A}\,\varphi\varphi' + \frac{2\sqrt{2}}{A^2} = 0.$$

Integrating it once, we arrive at the following Riccati equation:

$$\varphi' + \frac{1}{A\sqrt{2}}\,\varphi^2 + \frac{2\sqrt{2}}{A^2}\,\lambda = K, \quad K = \text{const.} \tag{4.56}$$

Thus, the invariant solution for the operator $X_1 + X_4$ has the form (4.55), where the function $\varphi(\lambda)$ is defined by Eq. (4.56).

4.8.6 Invariant solution for the operator $X_1 - X_4$

Proceedings as in the case of the operator $X_1 + X_4$, one can verify that the invariant solution for the operator $X_1 - X_4$ has the form

$$w = \varphi(\lambda) - \sqrt{2}\,t, \quad \lambda = \ln x + \frac{A}{2}\,t^2, \tag{4.57}$$

where the function $\varphi(\lambda)$ is defined by the following Riccati equation:

$$\varphi' + \frac{1}{A\sqrt{2}}\,\varphi^2 - \frac{2\sqrt{2}}{A^2}\,\lambda = K, \quad K = \text{const.} \tag{4.58}$$

4.8.7 Invariant solution for the operator X_5

The reckoning shows (cf. Section 4.8.8) that the operator

$$X_5 = t^2\frac{\partial}{\partial t} + tx\ln x\frac{\partial}{\partial x} + \left(\frac{\sqrt{2}}{A}\ln x - tw\right)\frac{\partial}{\partial w}$$

has the invariants

$$\lambda = \frac{\ln x}{t}\,, \quad \varphi = tw - \frac{\sqrt{2}}{A}\ln x.$$

Letting $\varphi = \varphi(\lambda)$ and substituting the resulting expression

$$w = \frac{\sqrt{2}}{A}\frac{\ln x}{t} + \frac{\varphi(\lambda)}{t}$$

in Eq. (4.21) we obtain

$$\frac{A^2}{2t^3}\left[\varphi'' + \frac{\sqrt{2}}{A}\varphi\varphi'\right] = 0,$$

or

$$\varphi'' + \frac{\sqrt{2}}{A}\varphi\varphi' = \left(\varphi' + \frac{1}{A\sqrt{2}}\varphi^2\right)' = 0,$$

whence

$$\varphi' + \frac{1}{A\sqrt{2}}\varphi^2 = \text{const}.$$

Integration of this equation yields

$$\text{(i)} \qquad \varphi(\lambda) = \frac{A\alpha}{\sqrt{2}}\frac{\beta e^{\alpha\lambda} - 1}{\beta e^{\alpha\lambda} + 1}\,,$$

$$\text{(ii)} \qquad \varphi(\lambda) = \frac{A\alpha}{\sqrt{2}}\frac{\beta e^{\alpha\lambda} + 1}{\beta e^{\alpha\lambda} - 1}\,,$$

$$\text{(iii)} \qquad \varphi(\lambda) = \frac{A\alpha}{\sqrt{2}}\tan\left(\beta - \frac{\alpha}{2}\lambda\right),$$

where $\alpha, \beta = \text{const}$. Hence, X_5 provides the following the invariant solutions:

$$\text{(i)} \qquad w = \frac{\sqrt{2}}{At}\left[\ln x + \frac{A^2\alpha}{2}\frac{\beta e^{\alpha\lambda} - 1}{\beta e^{\alpha\lambda} + 1}\right], \quad \lambda = \frac{\ln x}{t}\,, \qquad (4.59)$$

$$\text{(ii)} \qquad w = \frac{\sqrt{2}}{At}\left[\ln x + \frac{A^2\alpha}{2}\frac{\beta e^{\alpha\lambda} + 1}{\beta e^{\alpha\lambda} - 1}\right],$$

$$\text{(iii)} \qquad w = \frac{\sqrt{2}}{At}\left[\ln x + \frac{A^2\alpha}{2}\tan\left(\beta - \frac{\alpha}{2}\lambda\right)\right].$$

4.8.8 Invariant solution for the operator $X_1 + X_5$

This case is useful for illustrating all steps in constructing invariant solutions based on a rather complicated symmetry. Therefore, I will give here detailed calculations.

In order to find the invariants for the operator

$$X_1 + X_5 = X_5 = (1+t^2)\frac{\partial}{\partial t} + tx\ln x\frac{\partial}{\partial x} + \left(\frac{\sqrt{2}}{A}\ln x - tw\right)\frac{\partial}{\partial w}\,,$$

we have to find two functionally independent first integrals of the characteristic system

$$\frac{dt}{1+t^2} = \frac{dx}{tx\ln x} = \frac{dw}{\frac{\sqrt{2}}{A}\ln x - tw} \tag{4.60}$$

of the equation

$$(X_1 + X_5)J(t,x,w) = 0$$

for the invariants. Rewriting the first equation of the characteristic system in the form

$$\frac{d\ln x}{\ln x} = \frac{tdt}{1+t^2} \equiv \frac{1}{2}\,\frac{d(1+t^2)}{1+t^2}$$

and integrating it we obtain the first integral

$$\ln(\ln x) = \ln(1+t^2) + \text{const.},$$

which is convenient to write in the form

$$\frac{\ln x}{\sqrt{1+t^2}} = \text{const.}$$

The left-hand side of this first integral provides one of the invariants:

$$\lambda = \frac{\ln x}{\sqrt{1+t^2}}\,\cdot$$

Let us integrate the second equation of the characteristic system (4.60),

$$\frac{dt}{1+t^2} = \frac{dw}{\frac{\sqrt{2}}{A}\ln x - tw}\,\cdot$$

We rewrite it in the form

$$\frac{dw}{dt} + \frac{tw}{1+t^2} = \frac{\sqrt{2}}{A}\,\frac{\ln x}{1+t^2}\,,$$

eliminate x by using the first integral found above, namely, replace $\ln x$ by $\lambda\sqrt{1+t^2}$, and obtain the following non-homogeneous linear first-order equation:

$$\frac{dw}{dt} + \frac{tw}{1+t^2} = \frac{\sqrt{2}}{A}\,\frac{\ln x}{1+t^2}\,\cdot$$

The method of variation of the parameter yields its general solution

$$w = \frac{\sqrt{2}}{A}\,\frac{t\ln x}{1+t^2} + \frac{\varphi}{\sqrt{1+t^2}}\,, \tag{4.61}$$

where φ is the constant of integration. Solving the above equation with respect to φ one obtains the second invariant

$$\varphi = \sqrt{1+t^2}\,w - \frac{\sqrt{2}}{A}\,\frac{t\ln x}{\sqrt{1+t^2}}\,\cdot$$

However, the last step is unnecessary. Indeed, we let $\varphi = \varphi(\lambda)$ directly in Eq. (4.61) and obtain the following candidates for the invariant solution:

$$w = \frac{\sqrt{2}}{A}\,\frac{t\ln x}{1+t^2} + \frac{\varphi(\lambda)}{\sqrt{1+t^2}}\,, \quad \lambda = \frac{\ln x}{\sqrt{1+t^2}}\,\cdot \tag{4.62}$$

According to the general theory of invariant solutions (Lie [79], Ovsyannikov [91]), the substitution of (4.62) in Eq. (4.21) will reduce Eq. (4.21) to an ordinary differential equation containing only $\lambda, \varphi(\lambda)$ and the derivatives φ', φ'' of $\varphi(\lambda)$. Let us proceed.

Eqs. (4.62) yield:

$$\lambda_t = -\frac{t\ln x}{(1+t^2)\sqrt{1+t^2}}\,, \quad \lambda_x = \frac{1}{x\sqrt{1+t^2}}$$

and

$$w_t = \frac{\sqrt{2}}{A}\,\frac{\ln x}{1+t^2} - \frac{2\sqrt{2}}{A}\,\frac{t^2\ln x}{(1+t^2)^2} - \frac{t\varphi}{(1+t^2)\sqrt{1+t^2}} - \frac{t\,\ln x\,\varphi'}{(1+t^2)^2}\,,$$

$$w_x = \frac{\sqrt{2}}{A}\,\frac{t}{x(1+t^2)} + \frac{\varphi'}{x(1+t^2)}\,,$$

$$w_{xx} = -\frac{\sqrt{2}}{A}\,\frac{t}{x^2(1+t^2)} - \frac{\varphi'}{x^2(1+t^2)} + \frac{\varphi''}{x^2(1+t^2)\sqrt{1+t^2}}\,\cdot$$

Accordingly, we have:

$$
\begin{aligned}
& w_t + \frac{A}{\sqrt{2}}\left(w + \frac{A}{\sqrt{2}}\right) x w_x + \frac{A^2}{2}\, x^2 w_{xx} \\
& = \frac{\sqrt{2}}{A}\,\frac{\ln x}{1+t^2} - \frac{2\sqrt{2}}{A}\,\frac{t^2 \ln x}{(1+t^2)^2} - \frac{t\varphi}{(1+t^2)\sqrt{1+t^2}} - \frac{t\,\ln x\,\varphi'}{(1+t^2)^2} \\
& + \left(\frac{t\,\ln x}{1+t^2} + \frac{A}{\sqrt{2}}\,\frac{\varphi}{\sqrt{1+t^2}} + \frac{A^2}{2}\right)\left(\frac{\sqrt{2}}{A}\,\frac{t}{1+t^2} + \frac{\varphi'}{1+t^2}\right) \\
& - \frac{A}{\sqrt{2}}\,\frac{t}{1+t^2} - \frac{A^2}{2}\,\frac{\varphi'}{1+t^2} + \frac{A^2}{2}\,\frac{\varphi''}{(1+t^2)\sqrt{1+t^2}} \\
& = \frac{A^2}{2(1+t^2)\sqrt{1+t^2}}\left[\varphi'' + \frac{\sqrt{2}}{A}\,\varphi\varphi' + \frac{2\sqrt{2}}{A^3}\,\frac{\ln x}{\sqrt{1+t^2}}\right].
\end{aligned}
$$

Replacing in the last line $\varphi\varphi'$ by $\frac{1}{2}(\varphi^2)'$ and noting that $\frac{\ln x}{\sqrt{1+t^2}} = \lambda$, we see that Eq. (4.21) reduces to the following ordinary differential equation for $\varphi(\lambda)$:

$$
\varphi'' + \frac{1}{A\sqrt{2}}\,(\varphi^2)' + \frac{2\sqrt{2}}{A^3}\,\lambda = 0.
$$

Integrating it once, we arrive at the following Riccati equation:

$$
\varphi' + \frac{1}{A\sqrt{2}}\,\varphi^2 + \frac{\sqrt{2}}{A^3}\,\lambda^2 = K, \quad K = \text{const.} \tag{4.63}
$$

Thus, the invariant solution for the operator $X_1 + X_5$ has the form (4.62), where the function $\varphi(\lambda)$ is defined by the Riccati equation (4.63).

4.8.9 Invariant solution for the operator $X_2 + X_5$

For the operator

$$
X_2 + X_5 = X_5 = t^2\frac{\partial}{\partial t} + x(1 + t\ln x)\frac{\partial}{\partial x} + \left(\frac{\sqrt{2}}{A}\,\ln x - tw\right)\frac{\partial}{\partial w}\,,
$$

we have the following characteristic system for calculating the invariants:

$$
\frac{dt}{t^2} = \frac{dx}{x(1+t\ln x)} = \frac{dw}{\frac{\sqrt{2}}{A}\,\ln x - tw}\,.
$$

Writing the first equation of this system in the form

$$
\frac{d\ln x}{1+t\ln x} = \frac{dt}{t^2}\,,
$$

we obtain the non-homogeneous linear first-order equation

$$\frac{d\ln x}{dt} = \frac{1}{t}\ln x + \frac{1}{t^2}\cdot$$

Solving this equation, we have

$$\ln x = -\frac{1}{2t} + \lambda t, \tag{4.64}$$

where λ is the constant of integration. Hence, we have found the following invariant:

$$\lambda = \frac{\ln x}{t} + \frac{1}{2t^2}\cdot$$

Now we write the second equation of the characteristic system in the form

$$\frac{dw}{dt} + \frac{w}{t} = \frac{\sqrt{2}}{A}\frac{\ln x}{t^2}$$

which upon replacing $\ln x$ by its expression (4.64) becomes:

$$\frac{dw}{dt} + \frac{w}{t} = \frac{\sqrt{2}}{A}\Big(\frac{\lambda}{t} - \frac{1}{2t^3}\Big).$$

Solving this linear first-order equation we have:

$$w = \frac{\sqrt{2}}{A}\Big(\lambda + \frac{1}{2t^2}\Big) + \frac{\varphi}{t},$$

where φ is the constant of integration. Replacing here λ by its expression given above and letting $\varphi = \varphi(\lambda)$, we obtain the following candidates for the invariant solution:

$$w = \frac{\sqrt{2}}{A}\left(\frac{\ln x}{t} + \frac{1}{t^2}\right) + \frac{\varphi(\lambda)}{t}, \quad \lambda = \frac{\ln x}{t} + \frac{1}{2t^2}\cdot \tag{4.65}$$

Substituting (4.65) in Eq. (4.21) we obtain the following equation:

$$\varphi'' + \frac{1}{A\sqrt{2}}(\varphi^2)' - \frac{2\sqrt{2}}{A^3} = 0$$

which, upon integrating, yields the Riccati equation (4.58),

$$\varphi' + \frac{1}{A\sqrt{2}}\varphi^2 - \frac{2\sqrt{2}}{A^3}\lambda = K, \quad K = \text{const.}$$

Thus, the invariant solution for the operator $X_2 + X_5$ has the form (4.65), where the function $\varphi(\lambda)$ is defined by the Riccati equation (4.58).

4.8.10 Invariant solution for the operator $X_2 - X_5$

The invariant solution for the operator $X_2 - X_5$ has the form

$$w = \frac{\sqrt{2}}{A}\left(\frac{\ln x}{t} - \frac{1}{t^2}\right) + \frac{\varphi(\lambda)}{t}, \quad \lambda = \frac{\ln x}{t} - \frac{1}{2t^2}, \tag{4.66}$$

where the function $\varphi(\lambda)$ is defined by the Riccati equation (4.56),

$$\varphi' + \frac{1}{A\sqrt{2}}\varphi^2 + \frac{2\sqrt{2}}{A^2}\lambda = K, \quad K = \text{const.}$$

4.8.11 Optimal system of invariant solutions

Summing up the results of Section 4.8 and noting that invariant solutions based on two-dimensional subalgebras are particular cases of those based on one-dimensional subalgebras, we can formulate the following theorem.

Theorem 6.8. Every invariant solution of Eq. (4.21) is given either by elementary functions or by solving a Riccati equation. The following solutions provide an optimal system of invariant solutions so that any invariant solution can be obtained from them by transformations of the 5-parameter group admitted by Eq. (4.21).

$$X_1: \qquad \text{(i)} \quad w = \frac{A\sqrt{2}}{K + \ln x}\,; \tag{4.67}$$

$$\text{(ii)} \quad w = \frac{A\alpha}{\sqrt{2}}\,\frac{\beta x^{\alpha}+1}{\beta x^{\alpha}-1}\,;$$

$$\text{(iii)} \quad w = \frac{A\alpha}{\sqrt{2}}\,\tan\Big(\beta - \frac{\alpha}{2}\,\ln x\Big).$$

$$X_3: \qquad w = \frac{\varphi(\lambda)}{\sqrt{t}}\,, \quad \lambda = \frac{\ln x}{\sqrt{t}}\,, \tag{4.68}$$

$$\text{where} \quad \varphi' + \frac{1}{A\sqrt{2}}\,\varphi^2 - \frac{1}{A^2}\,\lambda\varphi = K.$$

$$X_4: \qquad w = \frac{1}{t}\left(\frac{\sqrt{2}}{A}\,\ln x + K\right). \tag{4.69}$$

$$X_1 + X_4: \qquad w = \sqrt{2}\,t + \varphi(\lambda), \quad \lambda = \ln x - \frac{A}{2}\,t^2, \tag{4.70}$$

$$\text{where} \quad \varphi' + \frac{1}{A\sqrt{2}}\,\varphi^2 + \frac{2\sqrt{2}}{A^2}\,\lambda = K.$$

$$X_1 - X_4: \qquad w = \varphi(\lambda) - \sqrt{2}\,t, \quad \lambda = \ln x + \frac{A}{2}\,t^2, \tag{4.71}$$

$$\text{where} \quad \varphi' + \frac{1}{A\sqrt{2}}\,\varphi^2 - \frac{2\sqrt{2}}{A^2}\,\lambda = K.$$

$$X_5: \quad \text{(i)} \quad w = \frac{\sqrt{2}}{At}\left[\ln x + \frac{A^2\alpha}{2}\,\frac{\beta \mathrm{e}^{\alpha\lambda} - 1}{\beta \mathrm{e}^{\alpha\lambda} + 1}\right], \quad \lambda = \frac{\ln x}{t}, \tag{4.72}$$

$$\text{(ii)} \quad w = \frac{\sqrt{2}}{At}\left[\ln x + \frac{A^2\alpha}{2}\,\frac{\beta \mathrm{e}^{\alpha\lambda} + 1}{\beta \mathrm{e}^{\alpha\lambda} - 1}\right],$$

$$\text{(iii)} \quad w = \frac{\sqrt{2}}{At}\left[\ln x + \frac{A^2\alpha}{2}\,\tan\left(\beta - \frac{\alpha}{2}\,\lambda\right)\right].$$

$$X_1 + X_5: \quad w = \frac{\sqrt{2}}{A}\,\frac{t\ln x}{1+t^2} + \frac{\varphi(\lambda)}{\sqrt{1+t^2}}, \quad \lambda = \frac{\ln x}{\sqrt{1+t^2}}, \tag{4.73}$$

$$\text{where} \quad \varphi' + \frac{1}{A\sqrt{2}}\,\varphi^2 + \frac{\sqrt{2}}{A^3}\,\lambda^2 = K.$$

$$X_2 + X_5: \quad w = \frac{\sqrt{2}}{A}\left(\frac{\ln x}{t} + \frac{1}{t^2}\right) + \frac{\varphi(\lambda)}{t}, \quad \lambda = \frac{\ln x}{t} + \frac{1}{2t^2}, \tag{4.74}$$

$$\text{where} \quad \varphi' + \frac{1}{A\sqrt{2}}\,\varphi^2 - \frac{2\sqrt{2}}{A^2}\,\lambda = K.$$

$$X_2 - X_5: \quad w = \frac{\sqrt{2}}{A}\left(\frac{\ln x}{t} - \frac{1}{t^2}\right) + \frac{\varphi(\lambda)}{t}, \quad \lambda = \frac{\ln x}{t} - \frac{1}{2t^2}, \tag{4.75}$$

$$\text{where} \quad \varphi' + \frac{1}{A\sqrt{2}}\,\varphi^2 + \frac{2\sqrt{2}}{A^2}\,\lambda = K.$$

In these solutions, α, β and K are arbitrary constants.

4.8.12 How to use the optimal system of invariant solutions

Subjecting the optimal system of invariants solutions to the group transformations generated by the operators (4.26), one obtains all (regular) invariant solutions of Eq. (4.21). One can easily verify that the optimal system of invariant solutions (4.67)-(4.75) contains 24 parameters. Since each of the thirteen types of of solutions (4.67)-(4.75) will gain four group parameters after subjecting it to the transformations generated by the operators (4.26) (not five due to the invariance of the solution under consideration with respect to one operator), we see that the invariant solutions provide a wide class of exact solutions containing 76 parameters.

I will illustrate the method by means of the group transformations generated by the last operator from (4.26):

$$X_5 = t^2\frac{\partial}{\partial t} + tx\ln x\frac{\partial}{\partial x} + \left(\frac{\sqrt{2}}{A}\ln x - tw\right)\frac{\partial}{\partial w}\cdot$$

The Lie equations

$$\begin{aligned} \frac{d\bar{t}}{da} &= \bar{t}^2, & \bar{t}\big|_{a=0} &= t,\\ \frac{d\bar{x}}{da} &= \bar{t}\bar{x}\ln\bar{x}, & \bar{x}\big|_{a=0} &= x,\\ \frac{d\bar{w}}{da} &= \frac{\sqrt{2}}{A}\ln\bar{x} - \bar{t}\bar{w}, & \bar{w}\big|_{a=0} &= w, \end{aligned}$$

provide the following group transformations:

$$\bar{t} = \frac{t}{1-at}, \quad \bar{x} = \exp\left(\frac{\ln x}{1-at}\right),$$
$$\bar{w} = (1-at)w + \frac{\sqrt{2}}{A}a\ln x. \tag{4.76}$$

Note that the inverse transformation is obtained from (4.76) by exchanging the variables $\bar{t}, \bar{x}, \bar{w}$ with t, x, w and replacing the group parameter a by $-a$. Hence:

$$t = \frac{\bar{t}}{1+a\bar{t}}, \quad x = \exp\left(\frac{\ln\bar{x}}{1+a\bar{t}}\right),$$
$$w = (1+a\bar{t})\bar{w} - \frac{\sqrt{2}}{A}a\ln\bar{x}. \tag{4.77}$$

Let us apply the transformation (4.76) to the trivial invariant solution (4.51) obtained by using the operator X_2. Since Eq. (4.21) is invariant under transformation (4.76), let us write the solution (4.51) in the form $\bar{w} = K$. Using (4.76), we obtain

$$(1-at)w + \frac{\sqrt{2}}{A}a\ln x = K.$$

Upon solving this equation for w, we obtain the following new solution to Eq. (4.21):

$$w = \frac{1}{1-at}\left(K - \frac{\sqrt{2}}{A}a\ln x\right). \tag{4.78}$$

According to our construction, the solution (4.78) is invariant under the operator $\widetilde{X}_2$ obtained from X_2 by the transformation (4.76). Let us find $\widetilde{X}_2$.

Since the solution $\bar{w} = K$ is written in the variables $\bar{t}, \bar{x}, \bar{w}$, the operator X_2 leaving it invariant should also be written in these variables. Therefore we take it in the form

$$\overline{X}_2 = \bar{x}\frac{\partial}{\partial \bar{x}}$$

and denote by $\widetilde{X}_2$ its expression

$$\widetilde{X}_2 = \overline{X}_2(t)\frac{\partial}{\partial t} + \overline{X}_2(x)\frac{\partial}{\partial x} + \overline{X}_2(w)\frac{\partial}{\partial w} \tag{4.79}$$

written in terms of the variables t, x, w defined by Eqs. (4.77). We have $\overline{X}_2(t) = 0$ and

$$\overline{X}_2(x) = \bar{x}\frac{\partial x}{\partial \bar{x}} = \frac{x}{1 + a\bar{t}} = (1 - at)x\,, \quad \overline{X}_2(w) = -\frac{\sqrt{2}}{A}\,a$$

because $1 + a\bar{t} = (1 - at)^{-1}$. Therefore

$$\widetilde{X}_2 = (1 - at)x\frac{\partial}{\partial x} - \frac{\sqrt{2}}{A}\,a\frac{\partial}{\partial w} = x\frac{\partial}{\partial x} - \frac{a}{A}\Big(Atx\frac{\partial}{\partial x} + \sqrt{2}\,\frac{\partial}{\partial w}\Big).$$

Comparing the result with the operators (4.26) we conclude that the transformation generated by X_5 maps X_2 to the operator

$$\widetilde{X}_2 = X_2 - \frac{a}{A}\,X_4. \tag{4.80}$$

One can easily check that the solution (4.78) is invariant under the operator (4.80).

Applying the above procedure to the invariant solution (4.50)(ii) one can verify that the transformation (4.76) maps the solution (4.50) to the new solution

$$w = \frac{A\alpha}{\sqrt{2}\,(1 - at)}\,\frac{\beta\exp\big(\frac{\alpha \ln x}{1-at}\big) + 1}{\beta\exp\big(\frac{\alpha \ln x}{1-at}\big) - 1} - \frac{\sqrt{2}\,a}{A}\,\frac{\ln x}{1 - at}\,\cdot \tag{4.81}$$

The corresponding operator X_1 is transformed by Eq. (4.79) where X_2 is replaced by X_1, namely:

$$\begin{aligned}\widetilde{X}_1 &= \frac{\partial t}{\partial \bar{t}}\,\frac{\partial}{\partial t} + \frac{\partial x}{\partial \bar{t}}\,\frac{\partial}{\partial x} + \frac{\partial w}{\partial \bar{t}}\,\frac{\partial}{\partial w}\\ &= \frac{1}{(1 + a\bar{t})^2}\,\frac{\partial}{\partial t} - \frac{a \ln \bar{x}}{(1 + a\bar{t})^2}\,x\,\frac{\partial}{\partial x} + a\bar{w}\,\frac{\partial}{\partial w}\,\cdot\end{aligned}$$

Replacing here $\bar{t}, \bar{x}, \bar{w}$ by their expressions (4.76), we obtain:

$$\widetilde{X}_1 = (1-at)^2\frac{\partial}{\partial t} - a(1-at)x\ln x\frac{\partial}{\partial x} + a\Big[(1-at)w + \frac{\sqrt{2}}{A}\,a\ln x\Big]\frac{\partial}{\partial w} = \frac{\partial}{\partial t}$$

$$-\,a\Big[2t\frac{\partial}{\partial t} + x\ln x\frac{\partial}{\partial x} - w\frac{\partial}{\partial w}\Big] + a^2\Big[t^2\frac{\partial}{\partial t} + tx\ln x\frac{\partial}{\partial x} + \Big(\frac{\sqrt{2}}{A}\ln x - tw\Big)\frac{\partial}{\partial w}\Big].$$

Hence, the image $\widetilde{X}_1$ of the operator X_1 is presented by the following linear combination of X_1, X_3 and X_5 :

$$\widetilde{X}_1 = X_1 - a\,X_3 + a^2 X_5. \tag{4.82}$$

One can verify that (4.81) is the invariant solution under the operator $\widetilde{X}_1$.

The same procedure is applicable to the invariant solutions defined by solving Riccati equations. Starting, e.g. with the invariant solution (4.55) written in the form

$$\bar{w} = \sqrt{2}\,\bar{t} + \varphi(\bar{\lambda}), \quad \bar{\lambda} = \ln\bar{x} - \frac{A}{2}\,\bar{t}^2,$$

we arrive to the new solution

$$w = \sqrt{2}\Big[\frac{t}{(1-at)^2} - \frac{a}{A}\,\frac{\ln x}{1-at}\Big] + \frac{\varphi(\mu)}{1-at}\,, \quad \mu = \frac{\ln x}{1-at} - \frac{At^2}{2(1-at)^2}\,, \tag{4.83}$$

where $\varphi(\mu)$ is defined by the Riccati equation (4.56):

$$\frac{d\varphi}{d\mu} + \frac{1}{A\sqrt{2}}\,\varphi^2 + \frac{2\sqrt{2}}{A^2}\,\mu = K, \quad K = \text{const.} \tag{4.84}$$

Applying the previous procedure to the operator X_1+X_4 giving the invariant solution (4.55) we obtain that it is mapped into the operator

$$\widetilde{X}_1 + \widetilde{X}_4 = X_1 - a\,X_3 + X_4 + a^2 X_5 \tag{4.85}$$

and that (4.83) is the invariant solution with respect to the operator (4.85).

It is easier to find the transformations (4.80), etc. of operators by using Eqs. (4.37) and (4.36) instead of the more complicated transformation rule (4.79). Let us find, e.g. the transformations of the operators X_1 and X_2 given by Eqs. (4.37) and (4.36). The operator X_1 has the coordinate vector

$$l = (1,0,0,0,0).$$

It is mapped by (4.36) with $a_5 = a$ to the vector

$$\tilde{l} = (1,0,-a,0,a^2).$$

Substituting the coordinates of this vector in Eq. (4.37) we obtain

$$\widetilde{X}_1 = X_1 - a\,X_3 + a^2 X_5,$$

i.e. the operator (4.82). Likewise, the coordinate vector

$$l = (0, 1, 0, 0, 0)$$

for the operator X_2 is mapped by (4.36) to the coordinate vector

$$\tilde{l} = (0, 1, 0, -\frac{a}{A}\,, 0)$$

of the operator (4.80):

$$\widetilde{X}_2 = X_2 - \frac{a}{A}\,X_4.$$

Furthermore, the coordinate vector

$$l = (1, 0, 0, 1, 0)$$

for the operator $X_1 + X_4$ is mapped by (4.36) with $a_5 = a$ to the coordinate vector

$$\tilde{l} = (1, 0, -a, 1, a^2)$$

of the operator (4.85):

$$\widetilde{X}_1 + \widetilde{X}_4 = X_1 - a\,X_3 + X_4 + a^2 X_5.$$

5 Invariant solutions of Burgers' equation

5.1 Optimal system of subalgebras

We will write the Burgers equation (4.18) in the form

$$u_t - u u_x - u_{xx} = 0 \tag{5.1}$$

and its symmetries (4.23) in the form

$$X_1 = \frac{\partial}{\partial t}\,, \quad X_2 = \frac{\partial}{\partial x}\,, \quad X_3 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} - u\frac{\partial}{\partial u}\,, \tag{5.2}$$

$$X_4 = t\frac{\partial}{\partial x} - \frac{\partial}{\partial u}\,, \quad X_5 = t^2\frac{\partial}{\partial t} + tx\frac{\partial}{\partial x} - (x + tu)\frac{\partial}{\partial u}\,\cdot$$

The commutator table of the five-dimensional Lie algebra L_5 spanned by the operators (5.2) has the form (4.27) with $A = 1$:

	X_1	X_2	X_3	X_4	X_5
X_1	0	0	$2X_1$	X_2	X_3
X_2	0	0	X_2	0	X_4
X_3	$-2X_1$	$-X_2$	0	X_4	$2X_5$
X_4	$-X_2$	0	$-X_4$	0	0
X_5	$-X_3$	$-X_4$	$-2X_5$	0	0

(5.3)

Therefore an optimal system of one-dimensional subalgebras of the Lie algebra L_5 of the symmetries of the Burgers equation is given by (4.38),

$$\begin{aligned} &X_1, \quad X_2, \quad X_3, \quad X_4, \quad X_1+X_4, \quad X_1-X_4,\\ &X_5, \qquad X_1+X_5, \qquad X_2+X_5, \qquad X_2-X_5, \end{aligned} \tag{5.4}$$

with the operators $X_1, \ldots, X_5$ given by (4.23).

5.2 Optimal system of invariant solutions

If we find the invariant solution for each operator of the optimal system (5.4) of one-dimensional subalgebras we will obtain an optimal system of invariant solutions for the Burgers equation (5.1). For the sake of simplicity, we will consider the positive values of the time t.

Note that the invariance condition for the operator X_2 from the optimal system (5.4) yields $u = u(t)$ and provides the trivial solution $u = \text{const}$. Therefore, one needs to construct the invariant solutions using only the remaining nine operators from (5.4). I will illustrate the construction of invariant solutions by using the operator X_3.

The operator

$$X_3 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} - u\frac{\partial}{\partial u}$$

has the invariants

$$\lambda = \frac{x}{\sqrt{t}}\,, \quad \varphi = \sqrt{t}\,u.$$

Therefore the candidates for the invariant solutions have the form

$$u = \frac{\varphi(\lambda)}{\sqrt{t}}\,, \quad \lambda = \frac{x}{\sqrt{t}}\,\cdot \tag{5.5}$$

The reckoning shows that

$$u_t = -\frac{1}{2t\sqrt{t}}\Big(\varphi + \lambda\varphi'\Big), \quad u_x = \frac{1}{t}\,\varphi', \quad u_{xx} = \frac{1}{t\sqrt{t}}\,\varphi''.$$

Substituting (5.5) and the above expressions for the derivatives in Eq. (5.1) and multiplying by $-t\sqrt{t}$ we obtain the second-order ordinary differential equation

$$\varphi'' + \varphi\varphi' + \frac{1}{2}\big(\varphi + \lambda\varphi'\big) = 0.$$

Writing it in the form

$$\varphi'' + \frac{1}{2}\big(\varphi^2\big)' + \frac{1}{2}\big(\lambda\varphi\big)' = 0$$

and integrating once, we obtain the following Riccati equation:

$$\varphi' + \frac{1}{2}\,\varphi^2 + \frac{1}{2}\,\lambda\varphi = K, \quad K = \text{const.} \tag{5.6}$$

Thus, the invariant solution for the operator X_3 has the form (5.5), where the function $\varphi(\lambda)$ is defined by the Riccati equation (5.6).

The operator

$$X_5 = t^2\frac{\partial}{\partial t} + tx\frac{\partial}{\partial x} - (x + tu)\frac{\partial}{\partial u}$$

has the invariants

$$\lambda = \frac{x}{t}\,, \quad \varphi = x + tu.$$

Therefore the candidates for the invariant solutions have the form

$$u = -\lambda + \frac{\varphi(\lambda)}{t}\,, \quad \lambda = \frac{x}{t}\,\cdot$$

The reckoning shows that

$$u_t = \frac{1}{t^2}\Big(x - \varphi - \lambda\varphi'\Big), \quad u_x = -\frac{1}{t} + \frac{1}{t^2}\,\varphi', \quad u_{xx} = \frac{1}{t^3}\,\varphi''.$$

Substituting these in Eq. (5.1) and multiplying by $-t^3$ we obtain the second-order ordinary differential equation

$$\varphi'' + \varphi\varphi' \equiv \Big(\varphi' + \frac{1}{2}\,\varphi^2\Big)' = 0.$$

Integrating it once, we obtain:

$$\frac{d\varphi}{d\lambda} + \frac{1}{2}\,\varphi^2 = \frac{C}{2}\,, \quad C = \text{const.}$$

Hence the function $\varphi(\lambda)$ is defined by the quadrature

$$\int \frac{d\varphi}{C - \varphi^2} = \frac{1}{2}\,\big(\lambda + K\big), \quad K = \text{const.}$$

Evaluating this integral separately for $C > 0,\ C < 0$ and $C = 0$ we obtain the functions $\varphi(\lambda)$ given further in Eqs. (5.12). Proceeding likewise with all operators from (5.4), we obtain the following optimal system of invariant solutions involving the arbitrary constants σ, γ and K.

$$X_1: \quad \text{(i)} \quad u = \frac{2}{x+\gamma}\,; \tag{5.7}$$

$$\text{(ii)} \quad u = \sigma\,\frac{\gamma e^{\sigma x}+1}{\gamma e^{\sigma x}-1} \equiv \tilde{\sigma}\tanh\left(\tilde{\gamma} + \frac{\tilde{\sigma}}{2}\,x\right);$$

$$\text{(iii)} \quad u = \sigma\,\tan\left(\gamma - \frac{\sigma}{2}\,x\right).$$

$$X_3: \quad u = \frac{\varphi(\lambda)}{\sqrt{t}}\,, \quad \lambda = \frac{x}{\sqrt{t}}\,, \tag{5.8}$$

$$\text{where} \quad \varphi' + \frac{1}{2}\,\varphi^2 + \frac{1}{2}\,\lambda\varphi = K.$$

$$X_4: \quad u = \frac{K-x}{t}\,\cdot \tag{5.9}$$

$$X_1 + X_4: \quad u = \varphi(\lambda) - t, \quad \lambda = x - \frac{t^2}{2}\,, \tag{5.10}$$

$$\text{where} \quad \varphi' + \frac{1}{2}\,\varphi^2 + \lambda = K.$$

$$X_1 - X_4: \quad u = t + \varphi(\lambda), \quad \lambda = x + \frac{t^2}{2}\,, \tag{5.11}$$

$$\text{where} \quad \varphi' + \frac{1}{2}\,\varphi^2 - \lambda = K.$$

$$X_5: \qquad u = -\lambda + \frac{\varphi(\lambda)}{t}, \quad \lambda = \frac{x}{t}, \quad \text{where} \tag{5.12}$$

$$\text{(i)} \quad \varphi(\lambda) = \sigma\,\frac{\gamma e^{\sigma\lambda} - 1}{\gamma e^{\sigma\lambda} + 1}, \quad |\varphi| < \sigma;$$

$$\text{(ii)} \quad \varphi(\lambda) = \sigma\,\frac{\gamma e^{\sigma\lambda} + 1}{\gamma e^{\sigma\lambda} - 1}, \quad |\varphi| > \sigma;$$

$$\text{(iii)} \quad \varphi(\lambda) = \sigma\,\tan\Big(\gamma - \frac{\sigma}{2}\,\lambda\Big);$$

$$\text{(iv)} \quad \varphi(\lambda) = \frac{2}{\lambda + \gamma}\cdot$$

$$X_1 + X_5: \qquad u = -\frac{tx}{1+t^2} + \frac{\varphi(\lambda)}{\sqrt{1+t^2}}, \quad \lambda = \frac{x}{\sqrt{1+t^2}}, \tag{5.13}$$

$$\text{where} \quad \varphi' + \frac{1}{2}\,\varphi^2 + \frac{1}{2}\,\lambda^2 = K.$$

$$X_2 + X_5: \qquad u = -\frac{x}{t} - \frac{1}{t^2} + \frac{\varphi(\lambda)}{t}, \quad \lambda = \frac{x}{t} + \frac{1}{2t^2}, \tag{5.14}$$

$$\text{where} \quad \varphi' + \frac{1}{2}\,\varphi^2 - \lambda = K.$$

$$X_2 - X_5: \qquad u = -\frac{x}{t} + \frac{1}{t^2} + \frac{\varphi(\lambda)}{t}, \quad \lambda = \frac{x}{t} - \frac{1}{2t^2}, \tag{5.15}$$

$$\text{where} \quad \varphi' + \frac{1}{2}\,\varphi^2 + \lambda = K.$$

Since invariant solutions based on subalgebras of higher dimensions are particular cases of those based on one-dimensional subalgebras, we arrive at the following result on invariant solutions of the Burgers equation.

Theorem 6.9. The solutions (5.7)-(5.15) provide an optimal system of invariant solutions so that all other invariant solutions are obtained from them by transformations of the group admitted by the Burgers equation. It follows that, every invariant solution of the Burgers equation is given either by elementary functions or by solving a Riccati equation.

5.3 Reductions to the Airy equation

The Riccati equations in (5.10), (5.11), (5.14) and (5.15) have the form

$$\varphi' + c_1\varphi^2 = c_2\lambda + c_3, \quad c_1, c_2, c_3 = \text{const.} \tag{5.16}$$

Here $\varphi' = d\varphi/d\lambda$. All equations (5.16) with $c_1 \neq 0,\ c_2 \neq 0$ can be reduced to the Airy equation and hence solved in terms of the Airy functions. Indeed, we first transform Eq. (5.16) to the form

$$\frac{d\psi}{d\mu} + \psi^2 = \mu \tag{5.17}$$

by scaling the variables φ and λ. Namely, setting $\varphi = k\psi, \quad \mu = l(c_2\lambda + c_3)$ we rewrite Eq. (5.16) as

$$\frac{d\psi}{d\mu} + \frac{kc_1}{lc_2}\,\psi^2 = \frac{\mu}{kl^2c_2}\,\cdot$$

Equating the coefficients for ψ^2 and μ to 1, i.e. letting

$$kc_1 = lc_2, \quad kl^2c_2 = 1,$$

and solving these equations for $k,\ l$, we obtain $k = c_1^{-2/3}\,c_2^{1/3},\ l = c_1^{1/3}\,c_2^{-2/3}$. Hence, the scaling

$$\varphi = c_1^{-2/3}\,c_2^{1/3}\psi, \quad \mu = c_1^{1/3}\,c_2^{-2/3}(c_2\lambda + c_3) \tag{5.18}$$

maps Eq. (5.16) into Eq. (5.17). The substitution

$$\psi = \frac{d\ln|z|}{d\mu} \equiv \frac{z'}{z}$$

reduces the Riccati equation (5.17) into the *Airy equation*

$$\frac{d^2z}{d\mu^2} - \mu\, z = 0. \tag{5.19}$$

The general solution to Eq. (5.19) is the linear combination

$$z = C_1\text{Ai}(\mu) + C_2\text{Bi}(\mu), \quad C_1, C_2 = \text{const.}, \tag{5.20}$$

of the *Airy functions* (see, e.g. [88], [97])

$$\begin{aligned} \text{Ai}(\mu) &= \frac{1}{\pi}\int_0^\infty \cos\Big(\mu\tau + \frac{1}{3}\,\tau^3\Big)d\tau,\\ \text{Bi}(\mu) &= \frac{1}{\pi}\int_0^\infty \Big[\exp\Big(\mu\tau - \frac{1}{3}\,\tau^3\Big) + \sin\Big(\mu\tau + \frac{1}{3}\,\tau^3\Big)\Big]d\tau. \end{aligned} \tag{5.21}$$

In the case of the solutions (5.10) and (5.14) we have the Riccati equation

$$\varphi' + \frac{1}{2}\varphi^2 = \lambda + K$$

with $c_1 = 1/2,\ c_2 = 1,\ c_3 = K$. Making the change of variables (5.18),

$$\varphi = 2^{2/3}\,\psi, \quad \mu = 2^{-1/3}\,(\lambda + K),$$

we arrive at Eq. (5.17). Hence, $\varphi(\lambda)$ in (5.11) and (5.14) is given by

$$\varphi(\lambda) = 2^{2/3}\,\frac{d}{d\lambda}\ln\bigl|C_1\mathrm{Ai}\bigl[2^{-1/3}\,(\lambda + K)\bigr] + C_2\mathrm{Bi}\bigl[2^{-1/3}\,(\lambda + K)\bigr]\bigr|. \quad (5.22)$$

In case of the solutions (5.10) and (5.15) we have the Riccati equation

$$\varphi' + \frac{1}{2}\varphi^2 = K - \lambda.$$

The change of variables (5.18) reducing it to the form (5.17) is writen

$$\varphi = -2^{2/3}\,\psi, \quad \mu = 2^{-1/3}\,(K - \lambda).$$

Hence, $\varphi(\lambda)$ in (5.10) and (5.15) is given by

$$\varphi(\lambda) = -2^{2/3}\,\frac{d}{d\lambda}\ln\bigl|C_1\mathrm{Ai}\bigl[2^{-1/3}\,(K - \lambda)\bigr] + C_2\mathrm{Bi}\bigl[2^{-1/3}\,(K - \lambda)\bigr]\bigr|. \quad (5.23)$$

5.4 Construction of all invariant solutions

Subjecting the optimal system of invariants solutions (5.7)-(5.15) to the group transformations generated by the operators (5.2), one obtains all regular invariant solutions of the Burgers equation. The solutions (5.7)-(5.15) involve 24 parameters. Moreover, each of the thirteen types of solutions (5.7)-(5.15) will gain four group parameters after subjecting them to the transformations generated by the operators (5.2) (not five due to the invariance of the solution under consideration with respect to one operator). Therefore after the group transformations of the optimal system of invariant solutions we will obtain a wide class of exact solutions containing 76 parameters.

The group transformations generated by the operators (5.2) can be easily found by solving the Lie equations. For example, for the operator X_5 the Lie equations are written

$$\frac{d\bar{t}}{da} = \bar{t}^{\,2}, \quad \frac{d\bar{x}}{da} = \bar{t}\,\bar{x}, \quad \frac{d\bar{u}}{da} = -(\bar{x} + \bar{t}\,\bar{u}).$$

Integrating them and using the initial conditions, $\bar{t} = t,\ \bar{x} = x,\ \bar{u} = u$ when $a = 0$, one obtains the following one-parameter transformation group:

$$\bar{t} = \frac{t}{1 - at}\,, \quad \bar{x} = \frac{x}{1 - at}\,, \quad \bar{u} = (1 - at)\,u - ax. \tag{5.24}$$

To illustrate the method for obtaining new solutions by subjecting the optimal system of invariant solutions to the group transformations, let us use the group transformation (5.24). The transformation (5.24) maps any known solution

$$u = \Phi(t, x)$$

of the Burgers equation to the following one-parameter set of new solutions:

$$u = \frac{ax}{1 - at} + \frac{1}{1 - at}\,\Phi\Big(\frac{t}{1 - at}\,, \frac{x}{1 - at}\Big). \tag{5.25}$$

Substituting in (5.25) the functions Φ given in the stationary solutions (5.7) we obtain the following non-stationary solutions (see Section 7.2.2 in [47]):

$$u = \frac{ax}{1 - at} + \frac{2}{x + \gamma(1 - at)}\,,$$

$$u = \frac{1}{1 - at}\left[ax + \tilde{\sigma}\,\mathrm{th}\left(\tilde{\gamma} + \frac{\tilde{\sigma}x}{2(1 - at)}\right)\right],$$

$$u = \frac{1}{1 - at}\left[ax + \sigma\,\mathrm{tg}\left(\gamma - \frac{\sigma x}{2(1 - at)}\right)\right].$$

Substituting in (5.25) the functions Φ given in the solution (5.8) we obtain the solution

$$u = \frac{ax}{1 - at} + \frac{\varphi(\lambda)}{(1 - at)\sqrt{1 - at}}\,, \quad \lambda = \frac{x + k - akt}{\sqrt{t(1 - at)}}\,,$$

where $\varphi(\lambda)$ satisfies the same Riccati equation as in (5.8):

$$\varphi' + \frac{1}{2}\,\varphi^2 + \frac{1}{2}\,\lambda\varphi = K.$$

Paper 7

Invariants of parabolic equations

Preprint of the paper [57]

Abstract. The article is devoted to the solution of the invariants problem for the one-dimensional parabolic equations written in the two-coefficient canonical form used recently by N.H. Ibragimov:

$$u_t - u_{xx} + a(t,x)u_x + c(t,x)u = 0.$$

A simple invariant condition is obtained for determining all equations that are reducible to the heat equation by the general group of equivalence transformations. The solution to the problem of invariants is given also in the one-coefficient canonical

$$u_t - u_{xx} + c(t,x)u = 0.$$

One of the main differences between these two canonical forms is that the equivalence group contains the *arbitrary* linear transformation of the dependent variable for the two-coefficient form and only a *special type* of the linear transformations of the dependent variable for the one-coefficient form contains.

1 Two-coefficient representation of parabolic equations

Any one-dimensional linear homogeneous parabolic equation can be reduced by an appropriate change of the independent variables to the following form:

$$u_t - u_{xx} + a(t,x)u_x + c(t,x)u = 0. \tag{1.1}$$

This form has been used in [53] and [52] (Paper 4 in this voluem) for obtaining a simple criteria for identifying the parabolic equations reducible to the heat equation by the *linear transformation of the dependent variable,* without changing the independent variables. The present paper is a continuation of the work [53] and contains, *inter alia*, a criteria (Theorem 1) for reducibility to the heat equation by the general group of equivalence transformations including changes of the independent variables.

1.1 Equivalence transformations

The equivalence algebra for Eq. (1.1) is spanned by the generator

$$Y_\sigma = \sigma u \frac{\partial}{\partial u} + 2\sigma_x \frac{\partial}{\partial a} + (\sigma_{xx} - \sigma_t - a\sigma_x)\frac{\partial}{\partial c} \tag{1.2}$$

of the usual linear transformation of the dependent variable and by the generators

$$\begin{aligned} Y_\alpha &= \alpha \frac{\partial}{\partial x} + \alpha' \frac{\partial}{\partial a}, \\ Y_\gamma &= 2\gamma \frac{\partial}{\partial t} + \gamma' x \frac{\partial}{\partial x} + (x\gamma'' - a\gamma')\frac{\partial}{\partial a} - c\gamma' \frac{\partial}{\partial c} \end{aligned} \tag{1.3}$$

of the corresponding transformations of the independent variables.

Here $\gamma = \gamma(t)$, $\alpha = \alpha(t)$ and $\sigma = \sigma(t, x)$ are arbitrary functions, the prime denotes the differentiation with respect to t.

1.2 Invariants

The semi-invariant obtained in [41] is written for Eq. (1.1) as follows [52]:

$$K = aa_x - a_{xx} + a_t + 2c_x. \tag{1.4}$$

The problem of invariants reduces to calculation of invariants of the form

$$J = J(K, K_x, K_t, K_{tt}, K_{xt}, K_{xx}, \ldots) \tag{1.5}$$

for the operators (1.3). The generators (1.2) and (1.3) become

$$\begin{aligned} Y_\sigma &= \sigma u \frac{\partial}{\partial u}, \quad Y_\alpha = \alpha \frac{\partial}{\partial x} + \alpha'' \frac{\partial}{\partial K}, \\ Y_\gamma &= 2\gamma \frac{\partial}{\partial t} + \gamma' x \frac{\partial}{\partial x} + (x\gamma''' - 3\gamma' K)\frac{\partial}{\partial K} \cdot \end{aligned} \tag{1.6}$$

We use generators in the prolonged form

$$
\begin{aligned}
X = & \xi\frac{\partial}{\partial t} + \eta\frac{\partial}{\partial x} + \zeta\frac{\partial}{\partial u} + \zeta^{K_t}\frac{\partial}{K_t} + \zeta^{K_x}\frac{\partial}{\partial K_x} \\
& + \zeta^{K_{tt}}\frac{\partial}{\partial K_{tt}} + \zeta^{K_{xt}}\frac{\partial}{\partial K_{xt}} + \zeta^{K_{xx}}\frac{\partial}{\partial K_{xx}} \\
& + \zeta^{K_{ttt}}\frac{\partial}{\partial K_{ttt}} + \zeta^{K_{xtt}}\frac{\partial}{\partial K_{xtt}} + \zeta^{K_{xxt}}\frac{\partial}{K_{xxt}} + \zeta^{K_{xxx}}\frac{\partial}{\partial K_{xxx}} + \dots
\end{aligned}
$$

and consider invariants of order N (the maximal order of the derivatives involved in the invariant). It is better to use the notation

$$
K_{kl} = \frac{\partial^{k+l}K}{\partial x^k \partial t^l}.
$$

Then the form of the above generator becomes

$$
X = \xi\frac{\partial}{\partial t} + \eta\frac{\partial}{\partial x} + \zeta\frac{\partial}{\partial u} + \zeta^{K}\frac{\partial}{\partial K} + \zeta^{K_t}\frac{\partial}{\partial K_t} + \zeta^{K_x}\frac{\partial}{\partial K_x} + \sum_{k,l}^{k+l\le N} \zeta^{K_{kl}}\frac{\partial}{\partial K_{kl}}\,.
$$

Let us begin with the operator Y_α. Its first three prolongations have the coefficients

$$
\begin{gathered}
\zeta^{K_t} = \alpha^{(3)} - K_x\alpha',\ \zeta^{K_x} = 0, \\
\zeta^{K_{tt}} = \alpha^{(4)} - K_x\alpha'' - 2K_{xt}\alpha',\ \zeta^{K_{xt}} = -K_{xx}\alpha',\ \zeta^{K_{xx}} = 0, \\
\zeta^{K_{ttt}} = \alpha^{(5)} - K_x\alpha^{(3)} - 3K_{xt}\alpha'' - 3K_{xtt}\alpha', \\
\zeta^{K_{xtt}} = -K_{xx}\alpha'' - 2K_{xxt}\alpha',\ \zeta^{K_{xxt}} = -K_{xxx}\alpha',\ \zeta^{K_{xxx}} = 0,
\end{gathered}
$$

and other prolongations of k-th order give

$$
\zeta^{K_{k0}} = 0,\ \ \zeta^{K_{0k}} = \alpha^{(k+2)} + a_k\alpha^{(k)} + \dots,\ \zeta^{K_{ls}} = b_s\alpha^{(s)} + \dots,\ (l \ge 1,\ l+s=k)
$$

with some functions a_k and b_s which do not depend on α and its derivatives.

Let us use the operator Y_γ,

$$
Y_\gamma = 2\gamma\frac{\partial}{\partial t} + \gamma'\,x\frac{\partial}{\partial x} + (x\gamma''' - 3\gamma' K)\frac{\partial}{\partial K}
$$

Its first and second prolongations are defined by the following coefficients:

$$
\begin{gathered}
\zeta^{K_t} = x\gamma^{(4)} - 3\gamma'' K - 5\gamma' K_t - \gamma'' x K_x, \\
\zeta^{K_x} = \gamma^{(3)} - 4\gamma' K_x,
\end{gathered}
$$

$$\zeta^{K_{tt}} = x\gamma^{(5)} - 3\gamma'''K - 8\gamma''K_t - \gamma'''xK_x - 7\gamma'K_{tt} - 2\gamma''xK_{xt},$$
$$\zeta^{K_{xt}} = \gamma^{(4)} - 4\gamma''K_x - 6\gamma'K_{xt} - \gamma''xK_{xx},$$
$$\zeta^{K_{xx}} = -5\gamma'K_{xx},$$

Notice that other prolongations give

$$\zeta^{K_{s0}} = -(s+3)\gamma', \quad (s \geq 3).$$

Let us assume that an invariant J does not depend on the derivatives K_{kl}, where $l \geq 1$. Then the equation $Y_\alpha J = 0$ is identically satisfied, and the equation $Y_\gamma J = 0$ becomes

$$-\gamma^{(3)}\frac{\partial J}{\partial K_x} + \gamma'\left(4K_x\frac{\partial J}{\partial K_x} + 5K_{xx}\frac{\partial J}{\partial K_{xx}} + \sum_{l\geq 3}(l+3)K_{l0}\frac{\partial J}{\partial K_{l0}}\right) = 0.$$

This gives that $\frac{\partial J}{\partial K_x} = 0$, and any solution of the equation

$$5K_{xx}\frac{\partial J}{\partial K_{xx}} + \sum_{l\geq 3}(l+3)K_{l0}\frac{\partial J}{\partial K_{l0}} = 0$$

gives an invariant. The characteristic system of this equation is

$$\frac{d\,K_{xx}}{5K_{xx}} = \frac{d\,K_{xxx}}{6K_{xxx}} = \frac{d\,K_{xxxx}}{7K_{xxxx}} = \ldots$$

Further we use the following statements that can be easily proved by using the above prolongations of the generators (1.6).

Lemma. The equation

$$K_{xx} = 0 \tag{1.7}$$

is invariant under the group of equivalence transformations with the generators (1.2) and (1.3).

Proof. Since the coefficients $\zeta^{K_{xx}}$ in the prolongations of the generators Y_σ and Y_α vanish, the transformations corresponding to these operators do not change the derivative K_{xx}. Furthermore, the transformation of K_{xx} corresponding to the generator Y_γ is defined by two terms:

$$Y_\gamma = 2\gamma\frac{\partial}{\partial t}\zeta^{K_{xx}} - 5\gamma'K_{xx}\frac{\partial}{\partial K_{xx}}\ldots$$

Hence, the transformation corresponding to the generator Y_γ just scales the derivative K_{xx} with some function depending on t. This also does not change the property (1.7). This completes the proof.

Theorem 1. Eq. (1.1) can be mapped to the heat equation by the general group of equivalence transformations with the generators (1.2), (1.3) if and only if the semi-invariant K of Eq. (1.1) satisfies the equation (1.7).

Proof. According to (1.2) and (1.3), the equivalence transformations comprise the linear transformation of the dependent variable:

$$v = V(x,t)\,u \tag{1.8}$$

with an arbitrary function $V(x,t) \neq 0$ and the change of the independent variables

$$\tau = H(t), \quad y = x\,\varphi_1(t) + \varphi_0(t), \tag{1.9}$$

where $\varphi_1(t) \neq 0$ and $H(t)$ is defined by the equation

$$H'(t) = \varphi_1^2(t). \tag{1.10}$$

The requirement that the transformations (1.8), (1.9) map the equation

$$v_\tau - v_{yy} + b_2(y,\tau)v_y + b_3(y,\tau)v = 0 \tag{1.11}$$

into the heat equation

$$u_t - u_{xx} = 0 \tag{1.12}$$

yields the following equations:

$$\begin{aligned} &2H'V_x\varphi_1 + (x\varphi_1' + \varphi_0' - H'b_2)\varphi_1^2 V = 0,\\ &H'V_{xx}\varphi_1 - H'V_x\varphi_1^2 b_2 - H'\varphi_1^3 b_3 V - V_t\varphi_1^3 + (x\varphi_1' + \varphi_0')V_x\varphi_1^2 = 0. \end{aligned} \tag{1.13}$$

The problem is to find the conditions for the coefficients $b_2(\tau,y)$, $b_3(\tau,y)$ that guarantee existence of the functions $V(t,x), H(t), \varphi_0(t)$ and $\varphi_1(t)$. To solve this problem, we have to investigate the compatibility of the overdetermined system of partial differential equations (1.13). To this end, we note that Eqs. (1.13) yield

$$\begin{aligned} V_x &= \frac{\varphi_1 V}{2H'}(-\varphi_0' - \varphi_1' x + H'b_2),\\ V_t &= \frac{V}{4H'\varphi_1}\left[-{\varphi_1'}^2\varphi_1 x^2 + 2\varphi_1'\varphi_1 x(H'b_2 - \varphi_0')\right.\\ &\quad \left. +\varphi_1(H'^2(2b_{2y} - b_2^2 - 4b_3) - {\varphi_0'}^2 + 2\varphi_0' H' b_2) - 2\varphi_1' H'\right]. \end{aligned}$$

Equating the mixed derivatives,

$$(V_t)_x = (V_x)_t,$$

we find

$$H'^{\,3}K = \varphi_0'' H' - \varphi_0' H'' + x(\varphi_1'' H' - \varphi_1' H'') = 0$$

or

$$K = \varphi_1^{-6}\left(y(\varphi_1''\varphi_1 - 2\varphi_1'^{\,2}) + \varphi_0''\varphi_1^2 - 2\varphi_0'\varphi_1'\varphi_1 - \varphi_1''\varphi_0\varphi_1 + 2\varphi_1'^{\,2}\varphi_0\right).$$

Thus, the semi-invariant K of the equations that are equivalent to the heat equation has to be linear with respect to y, i.e. satisfy the equation $K_{yy} = 0$.

Conversely, let us assume that the condition

$$K_{yy} = 0$$

is satisfied for Eq. (1.11), i.e.

$$K = y\phi_1(\tau) + \phi_0(\tau). \tag{1.14}$$

Choosing $H(t), \varphi_0(t)$ and $\varphi_1(t)$ satisfying the equations (1.9) and

$$\varphi_1'' = 2\frac{\varphi_1'^{\,2}}{\varphi_1} + \varphi_1^5\phi_1, \quad \varphi_0'' = 2\varphi_0'\,\frac{\varphi_1'}{\varphi_1} + \varphi_1^4(\phi_1\varphi_0 + \phi_0) \tag{1.15}$$

one can transform equation (1.11) into the heat equation. This completes the proof.

Example. Consider the equation

$$v_\tau - v_{yy} + \frac{y}{3\tau}\,v_y = 0. \tag{1.16}$$

It has the form (1.11) with

$$b_2 = \frac{y}{3\tau}\,, \quad b_3 = 0.$$

The semi-invariant (1.4) for Eq. (1.16) is

$$K = -\frac{2y}{9\tau^2}\,\cdot$$

It has the form (1.14) with

$$\phi_1 = -\frac{2}{9\tau^2}\,, \quad \phi_2 = 0. \tag{1.17}$$

Let us investigate reducibility of Eq. (1.16) to the heat equation (1.12) by the change of the independent variables (1.9) without changing the dependent variable, i.e. by letting $V = 1$ in (1.8). Then Eqs. (1.13) reduce to one equation

$$x\varphi_1' + \varphi_0' - H'b_2 = 0.$$

Substituting here the expression of b_2 and using Eqs. (1.9), (1.10), we obtain:

$$x\varphi_1'(t) + \varphi_0'(t) - \varphi_1^2\,\frac{x\varphi_1(t) + \varphi_0(t)}{3H(t)} = 0.$$

Upon separating the variables, this equations splits into two equations:

$$\varphi_1' - \frac{\varphi_1^3}{3H} = 0$$

and

$$\varphi_0' - \frac{\varphi_0\varphi_1^2}{3H} = 0.$$

We rewrite the first equation in the form

$$H(t) = \frac{\varphi_1^3(t)}{3\varphi_1'(t)} \tag{1.18}$$

and satisfy the second equation by letting $\varphi_0 = 0$. Differentiating (1.18) and using (1.10), we obtain

$$\frac{\varphi_1^3}{3\varphi_1'}\,\varphi_1'' = 0,$$

whence $\varphi_1'' = 0$. Thus,

$$\varphi_1(t) = pt + q.$$

We take for the simplicity $q = 0$ and obtain

$$\varphi_1(t) = p\,t, \quad p = \text{const.}$$

Then Eqs. (1.18), (1.9) and (1.17) yield:

$$\tau = H(t) = \frac{p^2t^3}{3}\,, \quad \phi_1 = -\frac{2}{p^4t^6}\,\cdot$$

Now one can readily verify that Eqs. (1.15) are satisfied for arbitrary p. We take $p = 1$ and obtain the following change of variables transforming Eq. (1.16) to the heat equation:

$$\begin{aligned} \tau &= \frac{1}{3}\,t^3, \\ y &= t\,x, \\ v &= u. \end{aligned} \tag{1.19}$$

Remark. In the case of three-coefficient representation of parabolic equations the test for reduction to the heat equation is more complicated, see, e.g. [69].

Further we will consider Eqs. (1.1) that are not equivalent to the heat equation. In other words, taking into account Lemma 2, we will assume that

$$K_{xx} \neq 0.$$

This assumption allows us to obtain the invariants

$$K_{l0} K_{xx}^{-(l+3)/5}, \quad (l \geq 3). \tag{1.20}$$

The reckoning shows the following operator is an invariant differentiation:

$$\mathcal{D} = \frac{1}{K_{xx}^{1/5}} D_x \tag{1.21}$$

Moreover, we can easily see that applying the invariant differentiation (1.21) to the invariant corresponding to $l = 3$:

$$J_* = \frac{K_{xxx}}{K_{xx}^{6/5}}\,, \tag{1.22}$$

we obtain all invariants (1.20). Hence, (1.22) provides a basis of these invariants.

Theorem 2. The operator (1.21),

$$\mathcal{D} = K_{xx}^{-1/5} D_x,$$

is an invariant differentiation.
Proof. Let J be a differential invariant, i.e.,

$$XJ = 0.$$

Recall that the coefficients of the generator X are

$$\xi = \alpha + \gamma' x, \quad \eta = 2\gamma, \quad \zeta^{K_{xx}} = -5\gamma' K_{xx}.$$

Notice also that

$$X(K_{xx}^{-1/5}) - K_{xx}^{-1/5} D_x \xi = 0. \tag{1.23}$$

Using the identity (see [92])

$$D_x(XF) = X(D_x F) + D_x \xi \, D_x F + D_x \eta \, D_t F$$

that holds for any function F one has

$$\begin{aligned} X(K_{xx}^{-1/5} D_x J) &= (D_x J) X(K_{xx}^{-1/5}) \\ &+ K_{xx}^{-1/5} \left(D_x(XJ) - D_x \xi \, D_x J - D_x \eta \, D_t J \right). \end{aligned} \tag{1.24}$$

Note that

$$XJ = 0$$

because J is an invariant, and

$$D_x \eta = 0$$

because

$$\eta = 2\gamma(t).$$

Therefore Eq. (1.24) becomes

$$X(K_{xx}^{-1/5} D_x J) = (X(K_{xx}^{-1/5}) - K_{xx}^{-1/5} D_x \xi)\, D_x J.$$

Now we use Eq. (1.23) and obtain the proof of the theorem:

$$X\big(K_{xx}^{-1/5} D_x(J)\big) = 0.$$

1.3 Fourth-order invariants

Splitting the equations $Y_\alpha J = 0$ and $Y_\gamma J = 0$ with respect to the functions α, γ, and their derivatives, one obtains that

$$J(K, K_x, K_{xx}, K_{xxx}, K_{xxt}, K_{xxxx}, K_{xxxt}, K_{xxtt})$$

and these equations are

$$\begin{gathered}
\frac{\partial J}{\partial K} - K_{xxx} \frac{\partial J}{\partial K_{xxtt}} = 0, \quad \frac{\partial J}{\partial K_x} - 5K_{xx} \frac{\partial J}{\partial K_{xxtt}} = 0,\\
7K_{xxxx} \frac{\partial J}{\partial K_{xxxx}} + 6K_{xxx} \frac{\partial J}{\partial K_{xxx}} + 5K_{xx} \frac{\partial J}{\partial K_{xx}} + 4K_x \frac{\partial J}{\partial K_x} + 3K \frac{\partial J}{\partial K}\\
+8K_{xxxt} \frac{\partial J}{\partial K_{xxxt}} + 7K_{xxt} \frac{\partial J}{\partial K_{xxt}} + 9K_{xxtt} \frac{\partial J}{\partial K_{xxtt}} = 0,\\
6K_{xxx} \frac{\partial J}{\partial K_{xxxt}} + 12K_{xxt} \frac{\partial J}{\partial K_{xxtt}} + 5K_{xx} \frac{\partial J}{\partial K_{xxt}} = 0,\\
K_{xxxx} \frac{\partial J}{\partial K_{xxxt}} + 2K_{xxxt} \frac{\partial J}{\partial K_{xxtt}} + K_{xxx} \frac{\partial J}{\partial K_{xxt}} = 0.
\end{gathered}$$

This system of equations is a complete system. Solving the first four equations, this system becomes

$$(5J_2 - 6J_1^2) \frac{\partial J}{\partial z_2} + 10 z_2 \frac{\partial J}{\partial z_1} = 0,$$

where $J(J_1, J_2, z_1, z_2)$ and

$$J_1 = K_{xx}^{-6/5} K_{xxx}, \quad J_2 = K_{xx}^{-7/5} K_{xxxx},$$
$$z_1 = K_{xx}^{-14/5} \left(K_{xx}(K_{xxtt} + KK_{xxx} + 5K_x K_{xx}) - \tfrac{6}{5} K_{xxt}^2 \right),$$
$$z_2 = K_{xx}^{-13/5} \left(K_{xx} K_{xxxt} - \tfrac{6}{5} K_{xxt} K_{xxx} \right).$$

Notice that

$$(5J_2 - 6J_1^2) = K_{xx}^{-12/5} \left(5K_{xx} K_{xxxx} - 6K_{xxx}^2 \right).$$

If it is assumed that

$$5K_{xxxx} K_{xx} - 6(K_{xxx})^2 \neq 0, \tag{1.25}$$

then the invariants are

$$J_1, J_2, (5J_2 - 6J_1^2) z_1 - 5z_2^2.$$

If one assumes that

$$5K_{xxxx} K_{xx} - 6(K_{xxx})^2 = 0, \tag{1.26}$$

then the invariants are defined by the equation

$$z_2 \frac{\partial J}{\partial z_1} = 0.$$

This means that: (a) if $z_2 = 0$, then the invariants are

$$J_1, J_2, z_1;$$

(b) if $z_2 \neq 0$, then the invariants are

$$J_1, J_2, z_2.$$

The invariants which do not depend on any conditions are only the invariants (1.20). Notice that the properties (1.25), (1.26) and $z_2 = 0$ or $z_2 \neq 0$ are invariant.

The obtained results comply with results of [58] on invariants of parabolic equations with three coefficients.

2 One-coefficient representation of parabolic equations

One can transform Eq. (1.1) to the equation (this representation was already known in the classical literature, see, e.g. [76])

$$u_t - u_{xx} + c(t,x)u = 0 \tag{2.27}$$

with one coefficient $c = c(x,t)$ by using the linear transformation

$$v = u\,\mathrm{e}^{\rho(x,t)}$$

of the dependent variable, where $\rho(t,x)$ is defined by (see Eq. (3.7) in [52])

$$2\rho + a(x,t) = 0.$$

2.1 Equivalence transformations

The equivalence group for Eq. (2.27) is defined by the generator

$$8\gamma\frac{\partial}{\partial t} + 4(\gamma' x + 2\alpha)\frac{\partial}{\partial x} + u(2\beta - \gamma'' x^2 - 4\alpha' x)\frac{\partial}{\partial u}$$
$$+(\gamma''' x^2 - 2\gamma'' + 4\alpha'' x - 2\beta' - 8\gamma' c)\frac{\partial}{\partial c}$$

spanned by the following three generators:

$$\begin{aligned}
X_3 &= 8\gamma\frac{\partial}{\partial t} + 4\gamma' x\frac{\partial}{\partial x} - u\gamma'' x^2\frac{\partial}{\partial u} + (\gamma''' x^2 - 2\gamma'' - 8\gamma' c)\frac{\partial}{\partial c},\\
X_2 &= 2\alpha\frac{\partial}{\partial x} - u\alpha' x\frac{\partial}{\partial u} + \alpha'' x\frac{\partial}{\partial c},\\
X_1 &= \beta u\frac{\partial}{\partial u} - \beta'\frac{\partial}{\partial c},
\end{aligned} \tag{2.28}$$

where γ, α and β are arbitrary functions of t.

We use the prolongations of the generator:

$$X = \xi\frac{\partial}{\partial t} + \eta\frac{\partial}{\partial x} + \zeta\frac{\partial}{\partial u} + \zeta^c\frac{\partial}{\partial c} + \zeta^{c_t}\frac{\partial}{\partial c_t} + \zeta^{c_x}\frac{\partial}{\partial c_x}$$
$$+\zeta^{c_{tt}}\frac{\partial}{\partial c_{tt}} + \zeta^{c_{xt}}\frac{\partial}{\partial c_{xt}} + \zeta^{c_{xx}}\frac{\partial}{\partial c_{xx}}$$
$$+\zeta^{c_{ttt}}\frac{\partial}{\partial c_{ttt}} + \zeta^{c_{xtt}}\frac{\partial}{\partial c_{xtt}} + \zeta^{c_{xxt}}\frac{\partial}{\partial c_{xxt}} + \zeta^{c_{xxx}}\frac{\partial}{\partial c_{xxx}} + \dots$$

It is better to use the notation

$$c_{kl} = \frac{\partial^{k+l} c}{\partial x^k \partial t^l}.$$

Then the form of the generator becomes

$$X = \xi\frac{\partial}{\partial t} + \eta\frac{\partial}{\partial x} + \zeta\frac{\partial}{\partial u} + \zeta^c\frac{\partial}{\partial c} + \zeta^{c_t}\frac{\partial}{\partial c_t} + \zeta^{c_x}\frac{\partial}{\partial c_x} + \sum_{k,l}\zeta^{c_{kl}}\frac{\partial}{\partial c_{kl}}\ .$$

2.2 Invariants

Let us use the operator X_1. Then the first prolongation is defined by the coefficients

$$\zeta^{c_t} = -\beta^{(2)},\ \zeta^{c_x} = 0,$$

and other prolongations of k-th order give

$$\zeta^{c_{0k}} = -\beta^{(k+1)}, \quad \zeta^{c_{ls}} = 0, \quad (l \geq 1,\ l+s=k).$$

Splitting with respect to the derivatives of the function $\beta(t)$ the equation for invariants

$$X_1^{(k)} J = 0,$$

we immediately come to the conditions that the invariant J does not depend on the derivatives

$$c_{0k},\ (k \geq 1).$$

Hence, further the coefficients $\zeta^{c_{0k}}$ are not necessary to be calculated. Splitting with respect to β' also excludes c from the invariant J. Let us use the operator X_2. Then the first and the second prolongations are defined by the coefficients:

$$\begin{aligned}
&\zeta^{c_x} = \alpha'',\\
&\zeta^{c_{xt}} = \alpha^{(3)} - 2\alpha' c_{xx},\\
&\zeta^{c_{xx}} = 0,\\
&\zeta^{c_{xtt}} = \alpha^{(4)} - 2\alpha'' c_{xx} - 4\alpha' c_{xxt},\\
&\zeta^{c_{xxt}} = -2\alpha' c_{xxx},\\
&\zeta^{c_{xxx}} = 0.
\end{aligned}$$

In the k-th prolongation the maximal order of the derivative $\alpha^{(k+1)}$ will be in the coefficient $\zeta^{c_{(k-1)1}}$. Hence, the invariant J does not depend on the derivatives

$$c_{(k-2)1},\ c_{(k-1)1}.$$

If one assumes that the invariant J does not depend on

$$c_{sl},\ (l \geq 1,\ s+l=k\geq 2),$$

then the term with α'' in the equation $X_2J=0$ gives

$$\frac{\partial J}{\partial c_x}=0.$$

Hence, the invariant can only depend on

$$c_{xx},\ c_{xxx},\ c_{xxxx},\ \dots, c_{k0}.$$

Let us use the operator X_3. The prolongations are defined by the coefficients:

$$\zeta^{c_x}=2x\gamma'''-12\gamma' c_x,\quad \zeta^{c_t}=x^2\gamma^{(4)}-2\gamma^{(3)}-4\gamma'' x c_x-8\gamma'' c-16\gamma' c_t,$$
$$\zeta^{c_{xx}}=2\gamma'''-16\gamma' c_{xx},\quad \zeta^{c_{xt}}=2x\gamma^{(4)}-4\gamma'' x c_{xx}-12\gamma'' c_x-20\gamma' c_{xt},$$
$$\zeta^{c_{30}}=-20\gamma' c_{xxx},\quad \zeta^{c_{40}}=-24\gamma' c_{40},\ \zeta^{c_{50}}=-28\gamma' c_{50},\ \dots$$

Splitting the equation $X_3J=0$ with respect to the derivative γ''', one obtains

$$\frac{\partial J}{\partial c_{xx}}=0.$$

The equation $X_3J=0$ becomes

$$5c_{xxx}\frac{\partial J}{\partial c_{xxx}}+\sum_{k\geq 4}(k+2)c_{k0}\frac{\partial J}{\partial c_{k0}}=0.$$

For solving this equation one needs to solve the characteristic system of equations

$$\frac{d\,c_{xxx}}{5c_{xxx}}=\frac{d\,c_{xxxx}}{6c_{xxxx}}=\frac{d\,c_{xxxxx}}{7c_{xxxxx}}=\dots$$

Note that the equations (2.27) with the coefficient $c(t,x)$ satisfying the condition $c_{xxx}=0$ are equivalent to the heat equation. Hence, for the equations that are not equivalent to the heat equation all invariants not depending on c_{kl} $(l\geq 1)$ are given by

$$c_{k0}c_{30}^{-(k+2)/5},\quad k\geq 4. \tag{2.29}$$

Thus, the basis of differential invariants (2.29) consists of the invariant

$$J=\frac{c_{xxxx}}{(c_{xxx})^{6/5}} \tag{2.30}$$

and the operator of invariant differentiation is

$$\mathcal{D} = c_{xxx}^{-1/5}\, D_x. \tag{2.31}$$

Proceeding as in the case of two coefficients, we arrive at the following statement.

Theorem 3. The operator $\mathcal{D}$ given by (2.31) is an operator of invariant differentiation. Hence, all invariants (2.29) are obtained from the invariant (2.30) by the invariant differentiation (2.31).

Paper 8

New identities connecting Euler-Lagrange, Lie-Bäcklund and Noether operators

N.H. Ibragimov, A.H. Kara and F.M. Mahomed [55]

Abstract. Euler-Lagrange, Lie-Bäcklund and Noether operators play a significant role in the study of symmetries and conservation laws of Euler-Lagrange equations. In this paper, we present fundamental operator identities involving these operators. Applications, e.g., to the inverse problem in the calculus of variations will be published elsewhere.

1 Notation and definitions

Let

$$x = (x^1, \ldots, x^n)$$

be the independent variable with coordinates x^i, and

$$u = (u^1, \ldots, u^m)$$

the dependent variable with coordinates u^α. The partial derivatives of u with respect to x are

$$u_i^\alpha = D_i(u^\alpha), \qquad u_{ij}^\alpha = D_j D_i(u^\alpha), \qquad \ldots, \tag{1.1}$$

where

$$D_i = \frac{\partial}{\partial x^i} + u_i^\alpha \frac{\partial}{\partial u^\alpha} + u_{ij}^\alpha \frac{\partial}{\partial u_j^\alpha} + \cdots, \quad i = 1, \ldots, n \tag{1.2}$$

is the operator of total differentiation. The collection of all first derivatives u_i^α is denoted by $u_{(1)}$. Similarly, the collections of all higher-order derivatives are denoted by $u_{(2)},\ u_{(3)}, \ldots$.

Denote by $\mathcal{A}$ the universal space of differential functions (i.e., locally analytic functions of a finite number of variables $x, u, u_{(1)}, u_{(2)}, \ldots$) introduced in [33] (see also [35], p. 56).

Definition 8.1. The *Euler-Lagrange* operator is defined by

$$\frac{\delta}{\delta u^\alpha} = \frac{\partial}{\partial u^\alpha} + \sum_{s\geq 1}(-1)^s D_{i_1}\cdots D_{i_s}\,\frac{\partial}{\partial u^\alpha_{i_1\cdots i_s}}\,, \quad \alpha = 1,\ldots,m\,. \tag{1.3}$$

The operator (1.3) is sometimes referred to as the *Euler operator*, named after Euler (1744) who first introduced it in a geometrical manner for the one-dimensional case. Also, it is called the *Lagrange operator*, bearing the name of Lagrange (1762) who considered the multi-dimensional case and established its use in a *variational* sense (see, e.g., [23] for a history of the calculus of variations). Following Lagrange, (1.3) is frequently referred to as a *variational derivative*. In the modern literature, the terminology Euler-Lagrange and variational derivative are used interchangeably as (1.3) usually arises in considering a variational problem.

Definition 8.2. The Lie-Bäcklund operator is given by the infinite formal sum

$$X = \xi^i\,\frac{\partial}{\partial x^i} + \eta^\alpha\,\frac{\partial}{\partial u^\alpha} + \zeta_i^\alpha\,\frac{\partial}{\partial u_i^\alpha} + \zeta^\alpha_{i_1i_2}\,\frac{\partial}{\partial u^\alpha_{i_1i_2}} + \cdots\,, \tag{1.4}$$

where $\xi^i,\ \eta^\alpha \in \mathcal{A}$ and the additional coefficients are determined uniquely by the prolongation formulae

$$\begin{aligned}
\zeta_i^\alpha \quad &= \quad D_i(W^\alpha) + \xi^j u^\alpha_{ij}\,,\\
\zeta^\alpha_{i_1i_2} \quad &= \quad D_{i_1}D_{i_2}(W^\alpha) + \xi^j u^\alpha_{ji_1i_2},\\
&\ldots\,.
\end{aligned} \tag{1.5}$$

In (1.5), W^α is the Lie characteristic function given by

$$W^\alpha = \eta^\alpha - \xi^j u_j^\alpha\,. \tag{1.6}$$

Sometimes it is convenient to write the Lie-Bäcklund operator (1.4) in the form

$$X = \xi^i D_i + W^\alpha\,\frac{\partial}{\partial u^\alpha} + D_i(W^\alpha)\,\frac{\partial}{\partial u_i^\alpha} + D_{i_1}D_{i_2}(W^\alpha)\,\frac{\partial}{\partial u^\alpha_{i_1i_2}} + \cdots \tag{1.7}$$

Noether [87] remarked that the finite transformation groups may depend on infinitely many derivatives if an infinitesimal transformation has derivatives of u. This was, in fact, covered by Bäcklund as is discussed in [32] (see also [37], Section 6.2.1).

Definition 8.3. Lie-Bäcklund operators X_1 and X_2 are said to be equivalent if $X_1 - X_2 = \lambda^i D_i, \quad \lambda^i \in \mathcal{A}$.

In modern group analysis, there exists a variety of so-called *generalised symmetries* which generalise Lie's point and contact infinitesimal group generators. However, the problem still remains whether these generalised symmetries generate, via the Lie equations, a group. The problem thus far is solved for Lie-Bäcklund operators (1.4). That is, the Lie equation is uniquely solvable, in the space $\big[[\mathcal{A}]\big]$ of formal power series with coefficients from $\mathcal{A}$ (the proof to be found in Ibragimov [32], Ibragimov [34]), for any Lie-Bäcklund operator (1.4). In this sense, Lie-Bäcklund symmetries are distinguished from all other generalised symmetries. Furthermore, the corresponding formal transformation group leaves invariant the contact conditions of any order. The possible existence of higher-order contact transformations were extensively discussed by S Lie and A V Bäcklund during the period 1874-1876. In recognition of their fundamental contributions, the above generalisation of Lie point and first-order contact transformations was given the name *Lie-Bäcklund transformations* by Ibragimov and Anderson [54]. The corresponding infinitesimal generator (1.4) is naturally called the *Lie-Bäcklund operator*. It should be noted that the prolongation formulae (1.5) are obtained as a direct consequence of the invariance of the infinite-order contact conditions.

Definition 8.4. The Noether operator (introduced in [31]) is defined by

$$N^i = \xi^i + W^\alpha \, \frac{\delta}{\delta u_i^\alpha} + \sum_{s\geq 1} D_{i_1} \cdots D_{i_s}(W^\alpha) \, \frac{\delta}{\delta u^\alpha_{ii_1\cdots i_s}} \, , \quad i = 1, \ldots, n, \quad (1.8)$$

where the Euler-Lagrange operators with respect to derivatives of u^α are obtained from (1.3) by replacing u^α by the corresponding derivatives, e.g.,

$$\frac{\delta}{\delta u_i^\alpha} = \frac{\partial}{\partial u_i^\alpha} + \sum_{s\geq 1} (-1)^s D_{j_1} \cdots D_{j_s} \, \frac{\partial}{\partial u^\alpha_{ij_1\cdots j_s}} \, , \quad (1.9)$$

where $i = 1, \ldots, n; \ \alpha = 1, \ldots, m$.

We recall that the famous Noether theorem [87] (see below) connecting symmetries and conservation laws for Euler-Lagrange equations is a direct consequence of the following operator identity (see [31], [34]).

The Euler-Lagrange, Lie-Bäcklund and Noether operators are connected by the operator identity

$$X + D_i(\xi^i) = W^\alpha \frac{\delta}{\delta u^\alpha} + D_i N^i \ . \tag{1.10}$$

The identity (1.10) is called the *Noether identity* in [31].

Definition 8.5. A Lie-Bäcklund operator X of the form (1.4) is called a *Noether symmetry* of a Lagrangian $L \in \mathcal{A}$ if there exists a vector

$$B = (B^1, \ldots, B^n), \quad B^i \in \mathcal{A},$$

such that

$$X(L) + L D_i(\xi^i) = D_i(B^i) \ . \tag{1.11}$$

Definition 8.6. If in (1.11)

$$B^i = 0, \quad i = 1, \ldots, n,$$

then X is referred to as a *strict Noether symmetry* of a Lagrangian $L \in \mathcal{A}$.

With the above definitions, Noether's theorem is formulated as follows.

For any Noether symmetry X of a given Lagrangian $L \in \mathcal{A}$, there corresponds a vector

$$T = (T^1, \ldots, T^n), \quad T^i \in \mathcal{A},$$

defined by

$$T^i = N^i(L) - B^i\,, \quad i = 1, \ldots, n, \tag{1.12}$$

which is a conserved vector of equation

$$\frac{\delta L}{\delta u^\alpha} = 0$$

i.e.

$$D_i(T^i) = 0$$

on the solutions of

$$\frac{\delta L}{\delta u^\alpha} = 0.$$

2 Any Noether symmetry is equivalent to a strict Noether symmetry

The following theorem is influenced by and generalizes an interesting example given by Noether in [87] (Section III).

Theorem 8.1. Any Noether symmetry is equivalent to a strict Noether symmetry. Namely, if the Lie-Bäcklund operator X satisfies Definition 8.5

$$X(L) + LD_i(\xi^i) = D_i(B^i) ,$$

then the equivalent operator

$$\tilde{X} = X - \frac{1}{L} B^i D_i .$$

is a strict Noether symmetry.

3 New identities: one-dimensional case

In the single independent variable case, we have the following theorems.

Theorem 8.2. The Lie-Bäcklund operator X can be represented in the form

$$X = ND . \tag{3.1}$$

Theorem 8.3. The following operator commutator relation holds:

$$[X, N] = ND(\xi) , \tag{3.2}$$

where

$$[X, N] = XN - NX .$$

4 New identities: multi-dimensional case

Theorem 8.4. The Lie-Bäcklund and Noether operators are related by the operator identity

$$\big[X + D_k(\xi^k), N^i\big] = D_k(\xi^i) N^k . \tag{4.1}$$

Remark 8.1. A more general result involving three Lie-Bäcklund operators X_1, X, X_2 such that $X_2 = [X, X_1]$ and the Noether operators N_1 and N_2 corresponding to X_1 and X_2 is presented in [70] (Section 2).

Theorem 8.5. The components of the conserved vector T^i, given by (1.12), associated with the Lie-Bäcklund operator X which is a generator of a Noether symmetry, satisfy

$$
\begin{aligned}
& X(T^i) + D_k(\xi^k)(T^i) - D_k(\xi^i)(T^k) \\
& = N^i\big(D_k(B^k)\big) + D_k(\xi^i)(B^k) - D_k(\xi^k)(B^i) - X(B^i). \qquad (4.2)
\end{aligned}
$$

Paper 9

Invariant Lagrangians

Unabridged draft of the paper [44]

Abstract. In this paper we will discuss a method for constructing Lagrangians for nonlinear differential equations using their Lie symmetries. We will find invariant Lagrangians and integrate second-order nonlinear ordinary differential equations admitting two-dimensional non-commutative Lie algebras. The method of integration of second-order equations suggested here is different from Lie's integration method based on canonical forms of two-dimensional Lie algebras. The new method reveals existence and significance of one-parameter families of singular solutions to nonlinear equations.

1 Introduction

Various problems of mathematics, mechanics, quantum field theory and other branches of theoretical and mathematical physics are connected with the calculus of variations. Often one deals with the *direct variational problem* when one starts with a given Lagrangian $L(x, u, u_{(1)})$ and investigates the dynamics of a physical system in question by considering the Euler-Lagrange equations. In the case of first-order Lagrangians $L(x, u, u_{(1)})$ the Euler-Lagrange equations are written (see the notation in Section 2)

$$\frac{\delta L}{\delta u^\alpha} = 0, \quad \alpha = 1, \ldots, m,$$

where

$$\frac{\delta L}{\delta u^\alpha} = \frac{\partial L}{\partial u^\alpha} - D_i\Big(\frac{\partial L}{\partial u_i^\alpha}\Big).$$

Example 9.1. Hamilton's principle of least action in classical mechanics states that a mechanical system is characterized by its Lagrangian

$$L(t, q, v) = T(t, q, v) - U(t, q),$$

where t is time, $q = (q^1, \ldots, q^m)$ and $v = dq/dt$ denote coordinates and velocities of particles of the system. $T(t, q, v)$ and $U(t, q)$ are the kinetic and potential energies of the system. The motion of the particles is determined by the Euler-Lagrange equations

$$\frac{\partial L}{\partial q^\alpha} - \frac{d}{dt}\frac{\partial L}{\partial v^\alpha} = 0, \quad \alpha = 1, \ldots, m.$$

In particular, let us consider a system consisting of a single particle with mass m =const. Let us define the position of a particle by its Cartesian coordinates $x = (x^1, x^2, x^3)$. The velocity of the particle is $v = (v^1, v^2, v^3)$ with $v^k = dx^k/dt$. The kinetic energy of the particle is equal to

$$T = \frac{m}{2}\sum_{k=1}^{3}(v^k)^2.$$

Hence the Lagrangian is

$$L = \frac{m}{2}\sum_{k=1}^{3}(v^k)^2 - U(t, x)$$

and the Euler-Lagrange equations provide the equations of motion

$$m\frac{d^2x^k}{dt^2} = -\frac{\partial U}{\partial x^k}, \quad k = 1, 2, 3.$$

For example, in Kepler's problem we deal with the motion of a planet in the potential field $U = \mu/r$ of the Sun, where $r = \sqrt{\sum_{k=1}^{3}(x^k)^2}$. Then

$$L = \frac{m}{2}\sum_{k=1}^{3}(v^k)^2 - \frac{\mu}{r}, \quad \mu = \text{const}.$$

Hence the equations of motion are written

$$m\frac{d^2x^k}{dt^2} = \mu\frac{x^k}{r^3}, \quad k = 1, 2, 3.$$

In the *inverse variational problem,* one starts with a given differential equation and looks for the corresponding Lagrangian. Sometimes, one can simply guess a Lagrangian, e.g. in the following examples.

Example 9.2. The simplest second-order ordinary differential equation

$$y'' = 0$$

has the Lagrangian

$$L = \frac{1}{2}\, y'^2$$

since it can be obtained as the Euler-Lagrange equation

$$\frac{\delta L}{\delta y} \equiv \frac{\partial L}{\partial y} - D_x\left(\frac{\partial L}{\partial y'}\right) = 0.$$

Indeed,

$$\frac{\partial L}{\partial y} = 0, \quad \frac{\partial L}{\partial y'} = y',$$

and hence

$$\frac{\delta L}{\delta y} = -D_x(y') = -y''.$$

Example 9.3. Likewise, one can easily see that the Laplace equation

$$\Delta u \equiv \sum_{i=1}^{n} \frac{\partial^2 u}{(\partial x^i)^2} = 0$$

has the Lagrangian

$$L = \frac{1}{2}\, |\nabla u|^2 \equiv \frac{1}{2} \sum_{i=1}^{n} \left(\frac{\partial u}{\partial x^i}\right)^2.$$

Example 9.4. Furthermore, the wave equation

$$u_{tt} - \Delta u = 0$$

has the Lagrangian

$$L = \frac{1}{2}\,\left(u_t^2 - |\nabla u|^2\right).$$

Example 9.5. One can verify that the Lin-Reissner-Tsien equation

$$2u_{tx} + u_x u_{xx} - u_{yy} = 0,$$

describing the transonic non-steady motion of a slender body in a compressible fluid, has the Lagrangian

$$L = -u_t u_x - \frac{1}{6} u_x^3 + \frac{1}{2} u_y^2.$$

Example 9.6. A less trivial example is provided by the equation of "short waves" from gas dynamics:

$$2u_{tx} - 2(x + u_x)u_{xx} + u_{yy} + 2ku_x = 0, \quad k = \text{const}.$$

It has the following Lagrangian (see [34], Section 23.4):

$$L = \left(xu_x^2 + \frac{1}{3} u_x^3 - u_t u_x - \frac{1}{2} u_y^2 \right) \mathrm{e}^{2(k+1)t}.$$

Indeed, the reckoning yields:

$$\frac{\delta L}{\delta u} \equiv \frac{\partial L}{\partial u} - D_t \left(\frac{\partial L}{\partial u_t} \right) - D_x \left(\frac{\partial L}{\partial u_x} \right) - D_y \left(\frac{\partial L}{\partial u_y} \right)$$

$$= \Big[2u_{tx} - 2(x + u_x)u_{xx} + u_{yy} + 2ku_x \Big] \mathrm{e}^{2(k+1)t}.$$

Example 9.7. Let us find the Lagrangian for an arbitrary linear second-order equation

$$y'' + a(x)y' + b(x)y = f(x).$$

Comparing with Example 9.2, we will seek the Lagrangian in the form

$$L = -\frac{p(x)}{2} y'^2 + \frac{q(x)}{2} y^2 - r(x)y.$$

Then

$$\frac{\delta L}{\delta y} = p(x)y'' + p'(x)y' + q(x)y - r(x).$$

Writing the Euler-Lagrange equation $\delta L / \delta y = 0$ in the form

$$y'' + \frac{p'(x)}{p(x)} y' + \frac{q(x)}{p(x)} y = \frac{r(x)}{p(x)}$$

and comparing with the second-order equation in question we obtain

$$p(x) = \mathrm{e}^{\int a(x)dx}, \quad q(x) = b(x)\,\mathrm{e}^{\int a(x)dx}, \quad r(x) = f(x)\,\mathrm{e}^{\int a(x)dx}.$$

Hence, the function

$$L = \left[-\frac{1}{2}\,y'^2 + \frac{b(x)}{2}\,y^2 - f(x)\,y \right] \mathrm{e}^{\int a(x)dx}$$

is the Lagrangian for our equation. Indeed, we have:

$$\frac{\delta L}{\delta y} = \left[y'' + a(x)y' + b(x)y - f(x) \right] \mathrm{e}^{\int a(x)dx}.$$

Note that not every differential equation, or a system of differential equations, has a Lagrangian. The heat equation

$$u_t - u_{xx} = 0$$

provides a simple example of an equation without a Lagrangian. Mathematical modelling provides many other equations having no Lagrangians. On the other hand, it is known ([9], [15]) that all second-order ordinary differential equations

$$y'' = f(x, y, y')$$

have Lagrangians. See further Theorem 9.1. The existence theorem does not furnish, however, simple practical devices for calculating Lagrangians.

The objective of the present paper is to show how to find Lagrangians for nonlinear second-order ordinary differential equations $y'' = f(x, y, y')$ with two known non-commuting symmetries and integrate the equation using invariant Lagrangians. Our main concern is on developing practical devises for constructing Lagrangians for non-linear equations rather than on theoretical investigation of solvability of the inverse variational problem. The method of *invariant Lagrangians* mentioned in [34], Section 25.3, is thoroughly illustrated by the following two nonlinear second-order ordinary differential equations:

$$y'' = \frac{y'}{y^2} - \frac{1}{xy}\,, \tag{1.1}$$

$$y'' = \mathrm{e}^y - \frac{y'}{x}\,\cdot \tag{1.2}$$

An attempt to find Lagrangians for Equations (1.1) and (1.2) by the "natural" approach employed in Example 9.7 shows conclusively that the inverse variational problem can be rather complicated even for ordinary differential equations. We will see that the concept of invariant Lagrangians allows one to find Lagrangians when the "guessing method" fails.

Furthermore, using equation (1.1) for illustration of Lie's integration method I came across the singular solutions

$$y = K\,x \qquad \text{and} \qquad y = \pm\sqrt{2x + C\,x^2}\,.$$

The new integration method presented here uncovers interesting connections of the singular solutions with singularities of the hypergeometric equation determining the Lagrangians. Moreover, the method furnishes an approach for obtaining one-parameter families of singular solutions.

2 Preliminaries

The Lagrangians in Examples 9.5 and 9.6 satisfy certain invariance property under the infinite groups admitted by the corresponding differential equations. Consequently, Noether's theorem allows one to obtain infinite sets of conservation laws (see, e.g. [34], Sections 23.3 and 23.4).

Furthermore, I used invariance properties for constructing Lagrangians for potential flows of compressible and incompressible fluids. In this way, new conservation laws were obtained in fluid dynamics ([29], see also [34], Sections 25.2 and 25.3). We will discuss invariant Lagrangians for potential flows of fluids in Section 2.4.

2.1 Derivative of nonlinear functionals

We use the usual notation from group analysis for independent variables x and dependent variables u together with their successive partial derivatives $u_{(1)},\, u_{(2)}, \ldots$. Namely:

$$x = \{x^i\},\ u = \{u^\alpha\},\ u_{(1)} = \{u_i^\alpha\},\ u_{(2)} = \{u_{ij}^\alpha\},\ \ldots,$$

where

$$u_i^\alpha = D_i(u^\alpha),\ u_{ij}^\alpha = D_i(u_j^\alpha) = D_i D_j(u^\alpha), \ldots\,,$$

$$(\alpha = 1, ..., m; \quad i,\, j, \ldots = 1, ..., n),$$

and D_i denote the total differentiations*:

$$D_i = \frac{\partial}{\partial x^i} + u_i^\alpha \frac{\partial}{\partial u^\alpha} + u_{ij}^\alpha \frac{\partial}{\partial u_j^\alpha} + u_{ijk}^\alpha \frac{\partial}{\partial u_{jk}^\alpha} + \cdots . \tag{2.1}$$

A locally analytic function $F(x, u, u_{(1)}, \ldots, u_{(s)})$ of a finite number of the variables $x, u, u_{(1)}, \ldots, u_{(s)}$ is called a *differential function* of order s.

An *action* (also termed a *variational integral*) used in Lagrangian mechanics is a nonlinear functional $l[u]$:

$$l[u] = \int_V L(x, u, u_{(1)}) \mathrm{d}x. \tag{2.2}$$

The integral is taken over an arbitrary n-dimensional domain V in the space of the independent variables x, and the integrand $L(x, u, u_{(1)})$ (termed *Lagrangian*) is a *first-order differential function*†.

The derivative $l'[u]$ of the functional (2.2) is a continuous linear functional $l'[u]\langle h \rangle$ in the Hilbert space L^2. Using the classical theorem of F. Riesz from functional analysis, one can represent the linear functional $l'[u]\langle h \rangle$ as the following scalar product in L^2 (see, e.g. [11]):

$$l'[u]\langle h \rangle = \left(\frac{\delta L}{\delta u^\alpha}, h \right) \equiv \int_V \left[\frac{\partial L}{\partial u^\alpha} - D_i \Big(\frac{\partial L}{\partial u_i^\alpha} \Big) \right] h^\alpha \mathrm{d}x,$$

where

$$\frac{\delta L}{\delta u^\alpha} = \frac{\partial L}{\partial u^\alpha} - D_i \left(\frac{\partial L}{\partial u_i^\alpha} \right) \tag{2.3}$$

is the *variational derivative.* The necessary condition $l'[u] = 0$ for extrema of the functional $l[u]$ provides the Euler-Lagrange equations:

$$\frac{\partial L}{\partial u^\alpha} - D_i \left(\frac{\partial L}{\partial u_i^\alpha} \right) = 0, \quad \alpha = 1, \ldots, m. \tag{2.4}$$

2.2 Invariance of functionals

Let G be a one-parameter group of transformations

$$\overline{x}^i = f^i(x, u, a), \quad \overline{u}^\alpha = \varphi^\alpha(x, u, a) \tag{2.5}$$

*Here and in what follows we employ the usual convention of summation in repeated indices.

†Restriction to first-order derivatives is not essential.

with the generator

$$X = \xi^i(x,u)\frac{\partial}{\partial x^i} + \eta^\alpha(x,u)\frac{\partial}{\partial u^\alpha}\cdot \tag{2.6}$$

Definition 9.1. The functional (2.2) is said to be invariant under the group G if the following equation holds for any domain V and any function $u(x)$:

$$\int_{\overline{V}} L(\overline{x},\overline{u},\overline{u}_{(1)})d\,\overline{x} = \int_V L(x,u,u_{(1)})dx. \tag{2.7}$$

Here $\overline{V}$ is the domain obtained from V by the transformation (2.5).

Lemma 9.1. The functional (2.2) is invariant under the group G with the generator (2.6) if and only if

$$X_{(1)}(L) + LD_i(\xi^i) = 0, \tag{2.8}$$

where $X_{(1)}$ denotes the first prolongation of the generator X,

$$X_{(1)}(L) = \xi^i\frac{\partial L}{\partial x^i} + \eta^\alpha\frac{\partial L}{\partial u^\alpha} + \zeta_i^\alpha\frac{\partial L}{\partial u_i^\alpha}\,, \quad \zeta_i^\alpha = D_i(\eta^\alpha) - u_j^\alpha D_i(\xi^j).$$

A simple proof of this statement is given in [28]. The proof is based on the fact that one can obtain the infinitesimal test for the invariance of nonlinear functionals $l[u]$ in terms of the differential n-form

$$Ldx \equiv Ldx^1 \wedge dx^2 \wedge \cdots \wedge dx^n \tag{2.9}$$

rather than of the integral (2.2). For details, see [34] or [39].

Let us distinguish Lagrangians of invariant functionals by the following definition.

Definition 9.2. The function $L(x,u,u_{(1)})$ satisfying Equation (2.8) is termed an *invariant Lagrangian* for the group G with the generator (2.10).

Remark 9.1. Invariant Lagrangians are, generally speaking, different from first-order *differential invariants* $F(x,u,u_{(1)})$ defined by the equation

$$X_{(1)}(F) = 0$$

instead of Equation (2.8).

2.3 Noether's theorem

Noether's theorem [87] states, e.g. in the case of first-order Lagrangians L and point transformation groups (2.5), that if the integral (2.2) is invariant under the group G with the generator (2.6),

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha} \tag{2.10}$$

then the quantities

$$A^i(x, u, u_{(1)}) = L\xi^i + \big(\eta^\alpha - \xi^j u_j^\alpha\big) \frac{\partial L}{\partial u_i^\alpha}\,, \quad i = 1, \ldots, n, \tag{2.11}$$

define a conserved vector

$$A = (A^1, \ldots, A^n)$$

for the Euler-Lagrange equations (2.4). In other words, the conservation equation

$$D_i\big(A^i\big) = 0 \tag{2.12}$$

holds on the solutions of Eqs. (2.4).

If the integral (2.2) is invariant under r linearly independent operators $X_1, \ldots, X_r$ of the form (2.10), then the formula (2.11) provides r linearly independent conserved vectors $A_1, \ldots, A_r$.

Noether's original proof was based on calculations involving variations of integrals $\int L\mathrm{d}x$. An alternative proof of Noether's theorem was given in [28] (see also [34] or [39] for more detailed presentation). The new approach allowed not only to simplify the proof of Noether's theorem, but also to get a new, more general theorem stating that the formulae (2.11) provide a conservation law for the Euler-Lagrange equations (2.4) if and only if the extremal values of the functional $l[u]$ are invariant under the group with the generator (2.10).

Remark 9.2. Conservation laws are obtained even when the invariance condition (2.8) is replaced by the *divergence relation*

$$X_{(1)}(L) + LD_i(\xi^i) = D_i(B^i), \tag{2.13}$$

where $B^i = B^i(x, u, u_{(1)})$. Then the conserved vector (2.11) is replaced by

$$A^i = L\xi^i + \big(\eta^\alpha - \xi^j u_j^\alpha\big) \frac{\partial L}{\partial u_i^\alpha} - B^i. \tag{2.14}$$

2.4 Invariant Lagrangians in fluid mechanics

Let us consider the equations of motion of an ideal incompressible fluid

$$\boldsymbol{v}_t + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v} + \nabla p = 0, \quad \operatorname{div}\boldsymbol{v} = 0, \tag{2.15}$$

where p is the pressure. The density of the fluid is set equal to $\rho = 1$.

In the case of potential flow one has

$$\boldsymbol{v} = \nabla\Phi, \tag{2.16}$$

and the system (2.15) yields the Laplace equation (due to $\operatorname{div}\boldsymbol{v} = 0$)

$$\Delta\Phi = 0 \tag{2.17}$$

and the Bernoulli integral

$$\Phi_t + \frac{1}{2}|\nabla\Phi|^2 + p = 0. \tag{2.18}$$

Here $\Phi = \Phi(t, \boldsymbol{x})$ is the potential and $\boldsymbol{x} = (x^1, \ldots, x^n)$, where $n \le 3$ is the dimension of the flow.

The variational derivative (2.3) is written

$$\frac{\delta L}{\delta\Phi} = \frac{\partial L}{\partial\Phi} - D_t\Big(\frac{\partial L}{\partial\Phi_t}\Big) - D_i\Big(\frac{\partial L}{\partial\Phi_i}\Big), \tag{2.19}$$

where $\Phi_i = \Phi_{x^i}$ whereas D_t and D_i denote the total differentiations (2.1) in t and in the spatial variables x^i $(i = 1, \ldots, n)$, respectively:

$$D_t = \frac{\partial}{\partial t} + \Phi_t\frac{\partial}{\partial\Phi} + \Phi_{tt}\frac{\partial}{\partial\Phi_t} + \Phi_{ti}\frac{\partial}{\partial\Phi_i} + \cdots\,,$$

$$D_i = \frac{\partial}{\partial x^i} + \Phi_i\frac{\partial}{\partial\Phi} + \Phi_{ti}\frac{\partial}{\partial\Phi_t} + \Phi_{ij}\frac{\partial}{\partial\Phi_j} + \cdots\,.$$

Using the Lagrangian from Example 9.6 for the Laplace equation and taking into account that

$$\frac{\delta\Phi_t}{\delta\Phi} = 0,$$

we will take the Lagrangian for Equation (2.17) in the form

$$L = \Phi_t + \frac{1}{2}\,|\nabla\Phi|^2. \tag{2.20}$$

Presence of the term Φ_t in the Lagrangian (2.20) and the concept of invariant Lagrangians allow us to obtain via Noether's theorem the classical and some non-classical conservation laws for incompressible fluids.

We will use the following notation. The generators (2.10) will be written

$$X = \xi^0 \frac{\partial}{\partial t} + \xi^i \frac{\partial}{\partial x^i} \eta \frac{\partial}{\partial \Phi} \cdot \tag{2.21}$$

Accordingly, the conservation equation (2.12) takes the form

$$D_t(A^0) + \nabla \cdot \boldsymbol{A} = 0. \tag{2.22}$$

The quantity A^0 and vector $\boldsymbol{A} = (A^1, \ldots, A^n)$ are given by Eqs. (2.11) written in the form

$$A^0 = L\xi^0 + W \frac{\partial L}{\partial \Phi_t}\,, \quad A^i = L\xi^i + W \frac{\partial L}{\partial \Phi_i}\,, \tag{2.23}$$

if the invariance condition (2.8) holds, and by Eqs. (2.14) written as

$$A^0 = L\xi^0 + W \frac{\partial L}{\partial \Phi_t} - B^0\,, \quad A^i = L\xi^i + W \frac{\partial L}{\partial \Phi_i} - B^i\,, \tag{2.24}$$

if the divergence relation (2.13) holds. We have used here the notation

$$W = \eta - \xi^0 \Phi_t - \xi^j \Phi_j. \tag{2.25}$$

After calculating conserved vectors, we will replace there $\nabla\Phi$ and Φ_t by $\boldsymbol{v}$ and p, respectively, using Eqs. (2.16) and (2.18). Furthermore, it is convenient for physical interpretation of conservation laws to write the conservation equation (2.22) in the form

$$D_t(\tau) + \operatorname{div}(\tau \boldsymbol{v} + \boldsymbol{\lambda}) = 0. \tag{2.26}$$

The standard procedure by using the divergence theorem allows one to rewrite the differential conservation law (2.26) in the integral form

$$\frac{d}{dt} \int\limits_{\Omega(t)} \tau d\omega = - \int\limits_{S(t)} (\boldsymbol{\lambda} \cdot \boldsymbol{\nu}) dS, \tag{2.27}$$

where $\Omega(t)$ is an arbitrary n-dimensional volume moving together with the fluid, $S(t)$ is its boundary, and $\boldsymbol{\nu}$ is the unit outer normal to $S(t)$.

One can find invariant Lagrangians and conservation laws starting with any symmetry of the Laplace equation.

Example 9.8. The time translation generator

$$X_1 = \frac{\partial}{\partial t}$$

satisfies the invariance condition (2.8). The quantity (2.25) is equal to

$$W = -\Phi_t$$

and Eqs. (2.23), (2.20) yield

$$A^0 = L - \Phi_t = \frac{1}{2}\,|\nabla\Phi|^2, \;\; A^i = -\Phi_t\Phi_i.$$

Hence,

$$A^0 = \frac{1}{2}\,|\nabla\Phi|^2, \quad \boldsymbol{A} = -\Phi_t\nabla\Phi.$$

Now we eliminate $\nabla\Phi$ and Φ_t by using Eqs. (2.16), (2.18) and obtain the conserved vector

$$A^0 = \frac{1}{2}\,|\boldsymbol{v}|^2, \quad \boldsymbol{A} = \left(\frac{1}{2}\,|\boldsymbol{v}|^2 + p\right)\boldsymbol{v}.$$

The corresponding conservation equation (2.22) has the form (2.26) with

$$\tau = \frac{1}{2}\,|\boldsymbol{v}|^2, \quad \boldsymbol{\lambda} = p\boldsymbol{v}.$$

Writing the conservation law in the integral form (2.27) we obtain the conservation of energy:

$$\frac{d}{dt}\int\limits_{\Omega(t)} \frac{1}{2}\,|\boldsymbol{v}|^2 d\omega = -\int\limits_{S(t)} p\,\boldsymbol{v}\cdot\boldsymbol{\nu} dS. \tag{2.28}$$

Remark 9.3. Note that the application of Noether's theorem to the potential flows gives us the conservation of energy (2.28) that is valid for all solutions of Eqs. (2.15).

Example 9.9. We will use now the dilation symmetry. By geometric reasons, it is convenient to take the infinitesimal dilation symmetry of the Laplace equation (2.17) in the following form (see [34], Section 10.2):

$$X = x^i\frac{\partial}{\partial x^i} - \frac{n-2}{2}\,\Phi\frac{\partial}{\partial\Phi}\,\cdot \tag{2.29}$$

Acting on the Lagrangian (2.20) by the first prolongation of Y,

$$X_{(1)} = x^i \frac{\partial}{\partial x^i} - \frac{n-2}{2}\,\Phi \frac{\partial}{\partial \Phi} - \frac{n-2}{2}\,\Phi_t \frac{\partial}{\partial \Phi_t} - \frac{n}{2}\,\Phi_i \frac{\partial}{\partial \Phi_i}\,, \tag{2.29$'$}$$

we obtain the following divergence relation (2.13):

$$X_{(1)}(L) + LD_i(\xi^i) = \frac{n+2}{2}\,\Phi_t = D_t\left(\frac{n+2}{2}\,\Phi\right).$$

Therefore, noting that the quantity (2.25) for the operator (2.29) is

$$W = -\frac{n-2}{2}\,\Phi - \boldsymbol{x}\cdot\nabla\Phi,$$

using Eqs. (2.14) [see also Eqs. (2.23)] and changing the sign of the resulting conserved vector, we have:

$$A^0 = n\Phi + \boldsymbol{x}\cdot\nabla\Phi,$$

$$A^i = \left(\frac{n-2}{2}\,\Phi + \boldsymbol{x}\cdot\nabla\Phi\right)v^i - \left(\Phi_t + \frac{1}{2}|\nabla\Phi|^2\right)x^i.$$

Hence, using Eqs. (2.16), (2.18) we obtain

$$A^0 = n\Phi + \boldsymbol{x}\cdot\boldsymbol{v}, \quad \boldsymbol{A} = \left(\frac{n-2}{2}\,\Phi + \boldsymbol{x}\cdot\boldsymbol{v}\right)\boldsymbol{v} + p\,\boldsymbol{x}. \tag{2.30}$$

Let us verify that (2.30) satisfies the conservation equation (2.22). Taking into account the second equation (2.15), $\operatorname{div}\boldsymbol{v} = 0$, one obtains after simple calculations:

$$D_t(A^0) + \nabla\cdot\boldsymbol{A} = \boldsymbol{x}\cdot[\boldsymbol{v}_t + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v} + \nabla p] + n[\Phi_t + \frac{1}{2}|\nabla\Phi|^2 + p].$$

Hence, the conservation equation (2.22) holds due to Eqs. (2.15), (2.18). Writing the conservation equation for (2.30) in the form (2.26) we obtain

$$\tau = n\Phi + \boldsymbol{x}\cdot\boldsymbol{v}, \quad \boldsymbol{\lambda} = p\,\boldsymbol{x} - \frac{n+2}{2}\,\Phi\boldsymbol{v}.$$

Hence the dilation symmetry (2.29) leads to the integral conservation law

$$\frac{d}{dt}\int\limits_{\Omega(t)} (n\Phi + \boldsymbol{x}\cdot\boldsymbol{v})d\omega = -\int\limits_{S(t)} \left(p\,\boldsymbol{x} - \frac{n+2}{2}\,\Phi\boldsymbol{v}\right)\cdot\boldsymbol{\nu}dS \tag{2.31}$$

for the potential flows of the incompressible fluid.

Example 9.10. Let us use the fact that the Lagrangian is not uniquely determined. We noticed in Example 9.9 that the Lagrangian (2.20) is not invariant with respect to the dilation symmetry (2.29). In order to find an invariant Lagrangian, we modify (2.20) as follows:

$$L = h(\boldsymbol{x})\Phi_t + \frac{q(t)}{2}\,|\nabla\Phi|^2, \quad q(t) \neq 0. \tag{2.32}$$

The variational derivative (2.19) of (2.32) is

$$\frac{\delta L}{\delta \Phi} = -q(t)\,\Delta\Phi. \tag{2.33}$$

Hence, the Euler-Lagrange equation for (2.32) with any functions $h(\boldsymbol{x})$ and $q(t) \neq 0$ gives again the Laplace equation (2.17). Furthermore, acting on (2.32) by the prolonged operator (2.29′) we obtain:

$$X_{(1)}(L) + LD_i(\xi^i) = \left(x^i\frac{\partial h}{\partial x^i} + \frac{n+2}{2}\,h\right)\Phi_t. \tag{2.34}$$

We can chose the coefficients $h(\boldsymbol{x})$ and $q(t)$ so that (2.32) will be an invariant Lagrangian with respect to the the dilation group with the generator (2.29). We will set $q(t) = 1$, i.e. take

$$L = h(\boldsymbol{x})\Phi_t + \frac{1}{2}\,|\nabla\Phi|^2 \tag{2.35}$$

and, according to Eq. (2.34), satisfy the invariance condition (2.8) by setting

$$x^i\frac{\partial h}{\partial x^i} + \frac{n+2}{2}\,h = 0.$$

Looking for a particular solution of the form $h = h(|\boldsymbol{x}|)$ of the above equation we reduce it to the linear ordinary differential equation

$$|\boldsymbol{x}|\frac{dh}{d|\boldsymbol{x}|} + \frac{n+2}{2}\,h = 0,$$

whence, ignoring the constant of integration, we get

$$h = |\boldsymbol{x}|^{-\frac{n+2}{2}}.$$

Thus, we have the following invariant Lagrangian:

$$L = |\boldsymbol{x}|^{-\frac{n+2}{2}}\Phi_t + \frac{1}{2}\,|\bigtriangledown\Phi|^2.$$

Applying Eqs. (2.11) to this Lagrangian and proceeding as in Example 9.10, one arrives at the following integral conservation law:

$$\frac{d}{dt}\int\limits_{\Omega(t)} |\boldsymbol{x}|^{-\frac{n+2}{2}}\left(\frac{n-2}{2}\,\Phi + \boldsymbol{x}\cdot\boldsymbol{v}\right)d\omega$$
$$= -\int\limits_{S(t)} |\boldsymbol{x}|^{-\frac{n+2}{2}}\left[\left(\frac{1}{2}|\boldsymbol{v}|^2 + p\right)\boldsymbol{x} - \left(\frac{n-2}{2}\,\Phi + \boldsymbol{x}\cdot\boldsymbol{v}\right)\boldsymbol{v}\right]\cdot \boldsymbol{\nu} dS. \tag{2.36}$$

Example 9.11. Let us find an invariant Lagrangian with respect to the dilation of $\boldsymbol{x}$ and arbitrary transformation of t with the generator

$$Y = x^i\frac{\partial}{\partial x^i} + g(t)\frac{\partial}{\partial t} \tag{2.37}$$

which is also admitted by the Laplace equation (2.17). Here $g(t)$ is an arbitrary function. We will search for an invariant Lagrangian in the form we modify (2.32) with $h(\boldsymbol{x}) = 0$, i.e. take

$$L = \frac{q(t)}{2}\,|\nabla\Phi|^2. \tag{2.38}$$

Substituting (2.37) and (2.38) in the left-hand side of the invariance condition (2.8) we obtain

$$Y_{(1)}(L) + (g'(t) + n)L = [(gq)' + (n-2)q]\frac{1}{2}\,|\nabla\Phi|^2.$$

Hence, the invariance of the Lagrangian (2.38) requires that

$$(gq)' + (n-2)q = 0.$$

Solving this linear homogeneous ordinary differential equation,

$$g(t)q' + (g'(t) + n - 2)q = 0,$$

we obtain

$$q = \frac{C}{g(t)}\,\mathrm{e}^{(2-n)\int\frac{dt}{g(t)}}, \quad C = \text{const}.$$

Setting $C = 2$ we the following invariant Lagrangian:

$$L = \frac{|\nabla\Phi|^2}{g(t)}\,\mathrm{e}^{-\int\frac{dt}{g(t)}}.$$

Example 9.12. The potential isentropic flow of a compressible fluid is described by the nonlinear partial differential equation

$$\Phi_{tt} + 2\nabla\Phi \cdot \nabla\Phi_t + \nabla\Phi \cdot (\nabla\Phi \cdot \nabla)\nabla\Phi + (\gamma - 1)\left(\Phi_t + \frac{1}{2}|\nabla\Phi|^2\right)\Delta\Phi = 0$$

considered together with the so-called Lagrange-Cauchy integral:

$$\Phi_t + \frac{1}{2}|\nabla\Phi|^2 + \frac{\gamma}{\gamma - 1}\rho^{\gamma-1} = 0,$$

where ρ is the density, Φ is the potential, and γ is a constant known in gas dynamics as the adiabatic exponent. Using the invariance under the group admitted by these equations the following Lagrangian was obtained [29]:

$$L = \left(\Phi_t + \frac{1}{2}|\nabla\Phi|^2\right)^{\frac{\gamma}{\gamma-1}}.$$

3 Lagrangians for second-order ODEs

3.1 Existence of Lagrangians

Definition 9.3. Definition 2. A differential function $L(x, y, y')$ is called a Lagrangian for a given second-order ordinary differential equation

$$y'' - f(x, y, y') = 0 \tag{3.1}$$

if equation (3.1) is equivalent to the Euler-Lagrange equation

$$\frac{\delta L}{\delta y} \equiv \frac{\partial L}{\partial y} - D_x\left(\frac{\partial L}{\partial y'}\right) = 0, \tag{3.2}$$

i.e. if $L(x, y, y')$ satisfies the equation

$$D_x\left(\frac{\partial L}{\partial y'}\right) - \frac{\partial L}{\partial y} = \sigma(x, y, y') \cdot [y'' - f(x, y, y')], \quad \sigma \neq 0, \tag{3.3}$$

with an undetermined multiplier $\sigma(x, y, y')$.

The expanded form of equation (3.3) is

$$y''L_{y'y'} + y'L_{yy'} + L_{xy'} - L_y = \sigma y'' - \sigma f(x, y, y').$$

Noting that the equality of the coefficients for y'' in both sides of the above equation yields $\sigma = L_{y'y'}$, we reduce equation (3.3) to the following *linear second-order partial differential equation* for unknown Lagrangians*:

$$f(x,y,y')L_{y'y'} + y'L_{yy'} + L_{xy'} - L_y = 0, \qquad L_{y'y'} \neq 0. \tag{3.4}$$

Definition 9.4. The inverse variational problem for the second-order ordinary differential equation (3.1) consists in finding a solution $L(x,y,y')$ of the partial differential equation (3.4) with the independent variables x, y, y' and the given function $f(x,y,y')$.

Theorem 9.1. Any second-order ordinary differential equation

$$y'' = f(x,y,y'),$$

where $f(x,y,y')$ is an arbitrary differential function, has a Lagrangian. In other words, the inverse variational problem has a solution for any second-order equation (3.1).

Proof. The proof follows almost immediately from the Cauchy-Kovalevski theorem. Let us first assume that $f(x,y,y') \neq 0$. Then the equation

$$f(x,y,y')\Omega_{y'}^2 + y'\Omega_y\Omega_{y'} + \Omega_x\Omega_{y'} = 0$$

for characteristics of the differential equation (3.4) is not satisfied by $\Omega = y'$, and hence the plane $y' = 0$ is not a characteristic surface. Consequently, the Cauchy-Kovalevski theorem guarantees existence of a solution to the Cauchy problem

$$f(x,y,y')L_{y'y'} + y'L_{yy'} + L_{xy'} - L_y = 0,$$

$$L\big|_{y'=0} = P(x,y), \quad L_{y'}\big|_{y'=0} = Q(x,y)$$

with arbitrary functions $P(x,y)$ and $Q(x,y)$. This solution satisfies the required condition $L_{y'y'} \neq 0$. Indeed, otherwise it would have the form $L = A(x,y)y' + B(x,y)$. Substitution of the latter expression in equation (3.4) shows that the functions $A(x,y)$ and $B(x,y)$ cannot be arbitrary, but should be restricted by the equation

$$\frac{\partial A(x,y)}{\partial x} = \frac{\partial B(x,y)}{\partial y}\,\cdot$$

*The requirement $L_{y'y'} \neq 0$, known as the *Legendre condition*, guarantees that the Euler-Lagrange equation (3.2), upon solving for y'', is identical with equation (3.1). See, e.g. [15], Chapter IV.

On the other hand, the initial conditions

$$L\big|_{y'=0} = P(x,y), \quad L_{y'}\big|_{y'=0} = Q(x,y)$$

require that the functions $A(x,y)$ and $B(x,y)$ should be equal to arbitrary functions $Q(x,y)$ and $P(x,y)$, respectively. Thus, equation (3.4) has solutions satisfying the Legendre condition $L_{y'y'} \neq 0$, provided that $f(x,y,y') \neq 0$. In the singular case $f = 0$ the existence of the required solution is evident since the equation $y'' = 0$ has the Lagrangian $L = y'^2/2$. This completes the proof (cf. [9], pp. 37-39; see also [15], Chapter IV, §12).

3.2 Concept of invariant Lagrangians

The above existence theorem does not furnish, however, simple practical devices for calculating Lagrangians of nonlinear differential equations (3.1). I suggested in 1983 ([34], Section 25.3, Remark 1) a method for determining invariant Lagrangians using the infinitesimal invariance test (2.8), and illustrated the efficiency of the method by second-order partial differential equations from fluid dynamics. I will give here a more detailed presentation of my method, develop a new integration theory based on invariant Lagrangians and illustrate it by non-linear ordinary differential equations of the second order.

In the case of ordinary differential equations, the invariance test (2.8) has the form

$$X_{(1)}(L) + D_x(\xi)L = 0. \tag{3.5}$$

We apply it to unknown Lagrangians $L(x,y,y')$ and known generators

$$X = \xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y}$$

admitted by an equation in question, where $\zeta(x,y,y')$ is obtained by the usual prolongation formula:

$$\zeta = D_x(\eta) - y'D_x(\xi) = \eta_x + (\eta_y - \xi_x)y' - y'^2\xi_y. \tag{3.6}$$

3.3 Integration using invariant Lagrangians

In the case of ordinary differential equations (3.1),

$$y'' = f(x,y,y'), \tag{3.7}$$

the infinitesimal symmetries (2.10) are written

$$X = \xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y} \tag{3.8}$$

and the conserved quantities (2.11) have the form

$$T = \xi L + \big(\eta - \xi y'\big)\frac{\partial L}{\partial y'}\,\cdot \tag{3.9}$$

The method of integration suggested here is quite different from Lie's classical methods (i.e. consecutive integration and utilization of canonical forms of two-dimensional Lie algebras, see, e.g. [39], Section 12.2) and comprises the following steps.

First step: Calculate the symmetries (3.8). Let equation (3.7) admit two linearly independent symmetries, X_1 and X_2.

Second step: Find an invariant Lagrangian $L(x,y,y')$ using the invariance test (3.5) under the operators X_1 and X_2 and then solving the defining equation (3.4) for L.

Third step: Use the invariant Lagrangian L and apply the formula (3.9) to the symmetries X_1 and X_2 to find two independent conservation laws:

$$T_1(x,y,y') = C_1, \quad T_2(x,y,y') = C_2. \tag{3.10}$$

Equations (3.10) mean only that the functions $T_1(x,y,y')$ and $T_2(x,y,y')$ preserve constant values along each solution of equation (3.7). However, if T_1 and T_2 are functionally independent, one can treat C_1 and C_2 as arbitrary parameters, since the Cauchy-Kovalevski theorem guarantees that equation (3.7) has solutions with any initial values of y and y', and hence C_1, C_2 in (3.10) can assume, in general, arbitrary values.

Fourth step: Eliminate y' from two equations (3.10) to obtain the solution of equation (3.7) in the implicit form:

$$F(x,y,C_1,C_2) = 0, \tag{3.11}$$

where C_1 and C_2 are two arbitrary parameters.

4 Invariant Lagrangians for Eq. (1.1)

We search for invariant Lagrangians for Equation (1.1),

$$y'' = \frac{y'}{y^2} - \frac{1}{xy}\,,$$

using the following two known symmetries ([39], Section 12.2.4):

$$X_1 = x^2 \frac{\partial}{\partial x} + xy \frac{\partial}{\partial y}\,, \quad X_2 = 2x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}\,\cdot \tag{4.1}$$

The determining equation (3.4) for the Lagrangians of equation (1.1) has the form

$$\left(\frac{y'}{y^2} - \frac{1}{xy}\right) L_{y'y'} + y' L_{yy'} + L_{xy'} - L_y = 0, \qquad L_{y'y'} \neq 0. \tag{4.2}$$

4.1 On the method of Euler-Laplace invariants

Recall that any linear hyperbolic second-order differential equations with two independent variables x, y :

$$a^{11} u_{xx} + 2a^{12} u_{xy} + a^{22} u_{yy} + b^1 u_x + b^2 u_y + cu = 0, \tag{4.3}$$

where $a^{11} = a^{11}(x,y), \ldots, c = c(x,y)$, can be rewritten in characteristic variables defined by the characteristic equation

$$a^{11} \Omega_x \, \Omega_x + 2a^{12} \Omega_x \, \Omega_y + a^{22} \Omega_y \, \Omega_y = 0, \tag{4.4}$$

in the standard form

$$L[u] \equiv u_{xy} + a(x,y) u_x + b(x,y) u_y + c(x,y) u = 0. \tag{4.5}$$

The *Euler-Laplace invariants* for Equation (4.5) are defined by*

$$h = a_x + ab - c, \quad k = b_y + ab - c. \tag{4.6}$$

Eq. (4.5) can be easily integrated by quadratures when one of the quantities (4.6) vanishes.

Namely, if $h = 0$ we rewrite the left-hand side of Eq. (4.5) in the form

$$L[u] = v_x + bv - hu, \quad \text{where} \quad v = u_y + au,$$

and arrive at the following integrable equation:

$$v_x + bv = 0.$$

**Author's note to this 2009 edition:* The quantities (4.6) are known in the literature as the Laplace invariants. I have learned from Louise Petrén's Thesis [95] that they have been discovered earlier by L. Euler. Therefore I will call them Euler-Laplace invariants.

Integration with respect to x yields

$$v = Q(y)\,\mathrm{e}^{-\int b(x,y)dx}$$

with an arbitrary function $Q(y)$. Substituting this expression in $u_y + au = v$, one obtains:

$$u_y + au = Q(y)\,\mathrm{e}^{-\int b(x,y)dx}$$

whence upon integration with respect to y :

$$u = \left[P(x) + \int Q(y)\mathrm{e}^{\int ady - bdx} dy\right] e^{-\int ady}. \tag{4.7}$$

Likewise, if $k = 0$ we rewrite the left-hand side of Eq. (4.5) in the form

$$Lu = w_y + aw - ku, \quad \text{where} \quad w = u_x + bu,$$

and obtain the following general solution of (4.5) with $k = 0$:

$$u = \left[Q(y) + \int P(x)e^{\int bdx - ady} dx\right] e^{-\int bdx}. \tag{4.8}$$

4.2 The Lagrangians admitting X_1

The invariance test (3.5) under the first generator (4.1),

$$X_1 = x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y}$$

provides the following non-homogeneous linear first-order partial differential equation:

$$x^2\frac{\partial L}{\partial x} + xy\frac{\partial L}{\partial y} + (y - xy')\frac{\partial L}{\partial y'} + 2xL = 0. \tag{4.9}$$

The implicit solution $V(x, y, y', L) = 0$ provides the homogeneous equation

$$x^2\frac{\partial V}{\partial x} + xy\frac{\partial V}{\partial y} + (y - xy')\frac{\partial V}{\partial y'} - 2xL\frac{\partial V}{\partial L} = 0. \tag{4.10}$$

The characteristic system for the latter equation,

$$\frac{dx}{x^2} = \frac{dy}{xy} = \frac{dy'}{y - xy'} = -\frac{dL}{2xL}$$

gives three first integrals:

$$\lambda = \frac{y}{x}\,, \quad \mu = y - xy', \quad \nu = x^2 L,$$

and the implicit solution $V(\lambda, \mu, \nu) = 0$ yields the solution

$$L = \frac{1}{x^2}\,\Phi(\lambda, \mu) \tag{4.11}$$

to the partial differential equation (4.9). We have by definition of λ and μ :

$$\lambda_x = -\frac{y}{x^2}\,, \quad \lambda_y = \frac{1}{x}\,, \quad \lambda_{y'} = 0, \quad \mu_x = -y, \quad \mu_y = 1, \quad \mu_{y'} = -x,$$

and therefore

$$L_{y'} = -\frac{1}{x}\,\Phi_\mu, \quad L_{y'y'} = \Phi_{\mu\mu}, \quad L_{yy'} = -\frac{1}{x^2}\,\Phi_{\lambda\mu} - \frac{1}{x}\,\Phi_{\mu\mu},$$

$$L_{xy'} = \frac{1}{x^2}\,\Phi_\mu + \frac{y}{x^3}\,\Phi_{\lambda\mu} + \frac{y'}{x}\,\Phi_{\mu\mu}, \quad L_y = \frac{1}{x^3}\,\Phi_\lambda + \frac{1}{x^2}\,\Phi_\mu\,\cdot$$

Substitution of these expressions reduces the equation (4.2) to the following linear equation with two variables λ and μ :

$$\mu\,\Phi_{\lambda\mu} - \frac{\mu}{\lambda^2}\,\Phi_{\mu\mu} - \Phi_\lambda = 0, \qquad \Phi_{\mu\mu} \neq 0. \tag{4.12}$$

The characteristics $\Omega(\lambda, \mu) = C$ of equation (4.12) are determined by the equation

$$\mu\,\Omega_\lambda \Omega_\mu - \frac{\mu}{\lambda^2}\,\Omega_\mu^2 \equiv \frac{\mu}{\lambda^2}\,(\lambda^2 \Omega_\lambda - \Omega_\mu)\,\Omega_\mu = 0$$

equivalent to the system of linear first-order equations

$$\Omega_\mu = 0, \qquad \lambda^2 \Omega_\lambda - \Omega_\mu = 0.$$

Two independent first integrals of the latter system have the form

$$\lambda = C_1, \qquad \mu - \frac{1}{\lambda} = C_2$$

and provide the characteristic variables u and v :

$$u = \lambda, \qquad v = \mu - \frac{1}{\lambda}\,\cdot$$

In these variables equation (4.12) takes the form

$$\Phi_{uv} - \frac{u}{1+uv}\,\Phi_u - \frac{1}{u(1+uv)}\,\Phi_v = 0. \tag{4.13}$$

i.e. the canonical form (4.5),

$$\Phi_{uv} + a(u,v)\,\Phi_u + b(u,v)\,\Phi_v + c(u,v)\Phi = 0 \tag{4.14}$$

with the coefficients

$$a(u,v) = -\frac{u}{1+uv}\,, \quad b(u,v) = -\frac{1}{u(1+uv)}\,, \quad c(u,v) = 0. \tag{4.15}$$

Equation (4.13) can be integrated by the method of Euler-Laplace invariants. Indeed, calculating the quantities (4.6),

$$h = a_u + ab - c, \qquad k = b_v + ab - c,$$

for Equation (4.13), one can readily see that one of these invariants vanishes, namely, $h = 0$. Therefore, one can obtain the solution to Equation (4.13) using the formula (4.7) for the general solution in the case $h = 0$:

$$\Phi(u,v) = \left[U(u) + \int V(v) \mathrm{e}^{\int a(u,v)dv - b(u,v)du}\, dv\right] \mathrm{e}^{-\int a(u,v)dv}, \tag{4.16}$$

where $U(u)$ and $V(v)$ are arbitrary functions.

Evaluating the integrals of the coefficients (4.15):

$$\int a(u,v)dv = -\ln|1+uv|, \quad \int b(u,v)du = -\ln\left|\frac{1+uv}{u}\right|$$

and using the formula (4.16) one obtains the following general solution to equation (4.13):

$$\Phi(u,v) = \Big[U(u) + \int V(v)\frac{udv}{(1+uv)^2}\Big](1+uv). \tag{4.17}$$

Let us take a particular solution, e.g. by letting $U(u) = 0,\ V(v) = v$ in (4.17). Then

$$\Phi = \frac{1}{u} + \frac{1+uv}{u}\ln|1+uv| = \frac{1}{\lambda} + \mu\ln|\lambda\mu| = \frac{x}{y} + (y - xy')\ln\left|\frac{y^2}{x} - yy'\right|,$$

and the formula (4.11) provides the following Lagrangian for equation (1.1):

$$L = \frac{1}{xy} + \left(\frac{y}{x^2} - \frac{y'}{x}\right) \ln\left|\frac{y^2}{x} - yy'\right|. \tag{4.18}$$

We have with this Lagrangian:

$$\frac{\delta L}{\delta y} = \frac{1}{xy' - y}\left(y'' - \frac{y'}{y^2} + \frac{1}{xy}\right).$$

Note that the exceptional situation in our approach that occurs when

$$y - xy' = 0,$$

singles out the solution $y = Cx$ to Equation (1.1). See also Remark 9.4 in Section 4.3.

4.3 The Lagrangians admitting X_1 and X_2

The prolongation of the second operator (4.1) is

$$X_2 = 2x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - y'\frac{\partial}{\partial y'}$$

hence the invariance test (3.5) under X_2 is written

$$2x\frac{\partial L}{\partial x} + y\frac{\partial L}{\partial y} - y'\frac{\partial L}{\partial y'} + 2L = 0. \tag{4.19}$$

By the same reasoning that led to equation (4.10), one obtains the equation

$$2x\frac{\partial V}{\partial x} + y\frac{\partial V}{\partial y} - y'\frac{\partial V}{\partial y'} - 2L\frac{\partial V}{\partial L} = 0. \tag{4.20}$$

Thus, the Lagrangians that are invariant under both X_1 and X_2 should solve simultaneously the equations (4.9) and (4.19):

$$\begin{aligned} x^2\frac{\partial L}{\partial x} + xy\frac{\partial L}{\partial y} + (y - xy')\frac{\partial L}{\partial y'} + 2xL &= 0,\\ 2x\frac{\partial L}{\partial x} + y\frac{\partial L}{\partial y} - y'\frac{\partial L}{\partial y'} + 2L &= 0. \end{aligned} \tag{4.21}$$

Looking for the solution in the implicit form $V(x, y, y', L) = 0$, one arrives at the system of two homogeneous equations (4.10) and (4.20):

$$
\begin{aligned}
Z_1(V) \equiv x^2\frac{\partial V}{\partial x} + xy\frac{\partial V}{\partial y} + (y - xy')\frac{\partial V}{\partial y'} - 2xL\frac{\partial V}{\partial L} &= 0,\\
Z_2(V) \equiv 2x\frac{\partial V}{\partial x} + y\frac{\partial V}{\partial y} - y'\frac{\partial V}{\partial y'} - 2L\frac{\partial V}{\partial L} &= 0,
\end{aligned}
\tag{4.22}
$$

where Z_1 and Z_2 are the following first-order linear differential operators:

$$
\begin{aligned}
Z_1 = x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y} + (y - xy')\frac{\partial}{\partial y'} - 2xL\frac{\partial}{\partial L}\,,\\
Z_2 = 2x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - y'\frac{\partial}{\partial y'} - 2L\frac{\partial}{\partial L}\,\cdot
\end{aligned}
\tag{4.23}
$$

The invariants for Z_1 are $\lambda = y/x,\ \mu = y - xy',\ \nu = x^2L$. Furthermore, we have $Z_2(\lambda) = -\lambda,\ Z_2(\mu) = \mu,\ Z_2(\nu) = 2\nu$, and hence

$$
Z_2 = -\lambda\frac{\partial}{\partial \lambda} + \mu\frac{\partial}{\partial \mu} + 2\nu\frac{\partial}{\partial \nu}\,\cdot
$$

The characteristic equations

$$
-\frac{d\lambda}{\lambda} = \frac{d\mu}{\mu} = \frac{d\nu}{2\nu}
$$

yield the following common invariants of Z_1 and Z_2 :

$$
z = \lambda\mu \equiv \frac{y^2}{x} - yy'\,, \quad q = \lambda^2\nu \equiv y^2L.
$$

Thus, the general solution of the system (4.22) is $V = V(z, q)$, and the equation $V(z, q) = 0$ provides the following solution to the system (4.21):

$$
L = \frac{1}{y^2}\Psi(z), \quad z = \frac{y^2}{x} - yy'\,\cdot
\tag{4.24}
$$

We have:

$$
z_x = -\frac{y\ 2}{x^2}\,, \quad z_y = 2\frac{y}{x} - y'\,, \quad z_{y'} = -y
$$

and therefore

$$
L_{y'} = -\frac{1}{y}\,\Psi'\,, \quad L_{y'y'} = \Psi''\,, \quad L_{yy'} = \frac{1}{y^2}\,\Psi' - \left(\frac{2}{x} - \frac{y'}{y}\right)\,\Psi''\,,
$$

$$L_{xy'} = \frac{y}{x^2}\,\Psi'', \quad L_y = -\frac{2}{y^3}\,\Psi + \left(\frac{2}{xy} - \frac{y'}{y^2}\right)\Psi'. \tag{4.25}$$

Substitution of these expressions reduces the equation (4.2) to a linear ordinary differential equation of the second order, namely to the hypergeometric equation (see Section 4.4)

$$z(1-z)\Psi'' + 2z\Psi' - 2\Psi = 0, \qquad \Psi'' \neq 0. \tag{4.26}$$

Equation (4.26) has singularities at points $z = 0,\ z = 1$ and the infinity, $z = \infty$. The singular points $z = 0$ and $z = 1$ define singular solutions to equations (1.1). This relationship between the singular points and singular solutions will be discussed in Section 5. Let us consider now the solutions to equation (4.26) at regular points z.

The substitution $\Psi = zw$ reduces (4.26) to the following integrable form discussed further in Section 4.4:

$$z(1-z)w'' + 2w' = 0. \tag{4.27}$$

Equation (4.27) can be solved in terms of elementary functions. Indeed, we have

$$\frac{dw'}{w'} = \frac{2dz}{z(z-1)} \equiv \frac{2dz}{z-1} - \frac{2dz}{z}$$

and obtain upon integration:

$$w' = C_1\Big(1 - \frac{1}{z}\Big)^2,$$

whence*

$$w = C_1\left(z - \frac{1}{z} - 2\ln|z|\right) + C_2. \tag{4.28}$$

We let $C_1 = 1, C_2 = 0$ and obtain the solution $\Psi = zw$ to Equation (4.26) in the following form:

$$\Psi(z) = z^2 - 1 - 2z\ln|z|. \tag{4.29}$$

Substituting (4.29) in (4.24), we obtain the following function $L(x, y, y')$:

$$L = -\frac{1}{y^2} + \frac{y^2}{x^2} - 2\frac{yy'}{x} + y'^2 - 2\Big(\frac{1}{x} - \frac{y'}{y}\Big)\ln\Big|\frac{y^2}{x} - yy'\Big|. \tag{4.30}$$

*The same result can be obtained from the general formula (4.41) with $\alpha = -2, \gamma = 0$.

Using (4.24), (4.25) and (4.29), one readily obtains

$$L_{y'} = \frac{2}{y} - 2\frac{y}{x} + 2y' + \frac{2}{y}\ln\left|\frac{y^2}{x} - yy'\right|, \qquad L_{y'y'} = 2 - \frac{2x}{x(y - xy'}\,,$$

$$L_{yy'} = -\frac{2}{x} + \frac{2}{y(y - xy')} - \frac{2}{y^2}\ln\left|\frac{y^2}{x} - yy'\right|, \qquad L_{xy'} = 2\frac{y}{x^2} - \frac{2}{x(y - xy'}\,,$$

$$L_y = \frac{2}{y^3} + 2\frac{y}{x^2} - 2\frac{y'}{x} - \frac{4}{xy} + 2\frac{y'}{y^2} - 2\frac{y'}{y^2}\ln\left|\frac{y^2}{x} - yy'\right|.$$

and hence the variational derivative of the function (4.30):

$$\frac{\delta L}{\delta y} = 2\frac{1-z}{z}\Big(y'' - \frac{y'}{y^2} + \frac{1}{xy}\Big) \equiv 2\frac{x - y^2 + xyy'}{y(y - xy')}\Big(y'' - \frac{y'}{y^2} + \frac{1}{xy}\Big).$$

Hence, the function $L(x, y, y')$ given by (4.30) is a Lagrangian for equation (1.1) with the exclusion of the singular points $z = 0$ and $z = 1$ of the hypergeometric equation (4.26).

Remark 9.4. According to (4.28) the general solution of equation (4.26) is given by

$$\Psi(z) = C_1(z^2 - 1 - 2z\ln|z|) + C_2 z.$$

It is spanned by the singular solution (4.29) and the regular solution $\Psi_* = z$. We eliminated the regular solution because it leads to the trivial Lagrangian (4.24), namely:

$$L_* = \frac{z}{y^2} \equiv \frac{1}{x} - \frac{y'}{y}\,\cdot$$

Its variational derivative $\delta L_*/\delta y$ vanishes identically.

4.4 Integration of a class of hypergeometric equations

In the theory of hypergeometric functions, the main emphasis is on asymptotics of the hypergeometric equation and its series solutions near the singular points (see, e.g. the classical book [106]). However, for our purposes we will need analytic expressions for the general solutions of certain types of the hypergeometric equation. Therefore, I determine in Theorem 9.2 a class of hypergeometric equations integrable by elementary functions or by quadrature. Numerous particular cases of this class can be found in various books on special functions, but to the best of my knowledge, not in the general form given in Theorem 9.2.

The second-order linear differential equation

$$x(1-x)\,y'' + [\gamma - (\alpha+\beta+1)x]\,y' - \alpha\beta\,y = 0 \tag{4.31}$$

with arbitrary parameters α, β, and γ is known as the *hypergeometric equation.* It has singularities at $x = 0,\ x = 1$ and $x = \infty$.

Furthermore, any homogeneous linear second-order differential equation of the form

$$(x^2 + Ax + B)\,y'' + (Cx + D)\,y' + E\,y = 0 \tag{4.32}$$

is transformable to the hypergeometric equation (4.31), provided that the equation

$$x^2 + Ax + B = 0$$

has two distinct roots x_1 and x_2. Indeed, rewriting equation (4.31) in the new independent variable t defined by

$$x = x_1 + (x_2 - x_1)t \tag{4.33}$$

one obtains

$$t(1-t)\,\frac{d^2y}{dt^2} + \left[\frac{Cx_1 + D}{x_1 - x_2} - C\,t\right]\frac{dy}{dt} - Ey = 0.$$

Setting here

$$\frac{Cx_1 + D}{x_1 - x_2} = \gamma, \quad C = \alpha + \beta + 1, \quad E = \alpha\beta$$

and then denoting the new independent variable t again by x, one arrives at Eq. (4.31).

If $\alpha\beta = 0$ the hypergeometric equation (4.31) is integrable by two quadratures. Indeed, letting, e.g. $\beta = 0$ and integrating the equation

$$\frac{dy'}{y'} = \frac{(\alpha+1)x - \gamma}{x(1-x)}\,dx,$$

we have

$$y' = C_1 \mathrm{e}^{q(x)},\ C_1 = \text{const.},$$

where

$$q(x) = \int \frac{(\alpha+1)x - \gamma}{x(1-x)}\,dx.$$

The second integration yields:

$$y = C_1 \int \mathrm{e}^{q(x)}\, dx + C_2, \quad C_1,\ C_2 = \text{const.}$$

The following theorem singles out the equations (4.31) with $\alpha\beta \neq 0$ that can be integrated by transforming them to equations not containing the term with w.

Theorem 9.2. The general solution of the hypergeometric equation (4.31) with $\beta = -1$ and two arbitrary parameters α and γ :

$$x(1-x)\, y'' + (\gamma - \alpha\, x)\, y' + \alpha\, y = 0 \tag{4.34}$$

is given by quadrature and has the form

$$y = C_1 \left(x - \frac{\gamma}{\alpha}\right) \int \left(|x|^{-\gamma}\, |x-1|^{\gamma-\alpha}\, [x - (\gamma/\alpha)]^{-2}\right) dx + C_2 \left(x - \frac{\gamma}{\alpha}\right), \tag{4.35}$$

where C_1 and C_2 are arbitrary constants.

Proof. Since the case $\alpha = 0$ was considered above, we assume in what follows that $\alpha \neq 0$. Let

$$y = \sigma(x)\, w. \tag{4.36}$$

Substitution of the expressions

$$y = \sigma(x)\, w, \quad y' = \sigma(z)\, w' + \sigma'(z)\, w,$$
$$y'' = \sigma(z)\, w'' + 2\sigma'(z)\, w' + \sigma''(z)\, w$$

in Eq. (4.31) yields:

$$\begin{aligned} &x(1-x)\sigma\, w'' + \{2x(1-x)\sigma' + [\gamma - (\alpha+\beta+1)x]\sigma\}\, w' \\ &+ \{x(1-x)\sigma'' + [\gamma - (\alpha+\beta+1)x]\, \sigma' - \alpha\beta\, \sigma\}\, w = 0. \end{aligned} \tag{4.37}$$

To annul the term with w, we have to find $\sigma(x)$ satisfying the equation

$$x(1-x)\sigma'' + [\gamma - (\alpha+\beta+1)x]\, \sigma' - \alpha\beta\, \sigma = 0. \tag{4.38}$$

It seems that we did not make any progress since we have to solve the original equation (4.31) for the unknown function $\sigma(x)$. However, we will take a particular solution $\sigma(x)$ by letting $\sigma''(x) = 0$. Hence, we consider the transformation (4.36) of the form

$$y = (kx + l)\, w, \quad k, l = \text{const.}$$

Then Eq. (4.38) reduces to

$$\gamma k - \alpha\beta l = 0, \quad k(\alpha + 1)(\beta + 1) = 0.$$

Since $\alpha\beta \neq 0$, it follows from the above equations that $k \neq 0$, and hence $\beta = -1$ (or $\alpha = -1$, but since equation (4.31) is symmetric with respect to the substitution $\alpha \leftrightarrows \beta$ we shall consider only $\beta = -1$). In what follows, we can set $k = 1$. Then the first equation of the above system yields $l = -\gamma/\alpha$. Thus we arrive at equation (4.34). Furthermore, it follows from (4.37) that the substitution

$$y = \Big(x - \frac{\gamma}{\alpha}\Big)w \tag{4.39}$$

reduces (4.34) to the equation

$$x(1-x)\Big(x - \frac{\gamma}{\alpha}\Big)\, w'' + \Big[2x(1-x) - \alpha\Big(x - \frac{\gamma}{\alpha}\Big)^2\Big]\, w' = 0. \tag{4.40}$$

One can readily integrate equation (4.40) in terms of elementary functions and one quadrature. Indeed, the equation

$$\frac{dw'}{w'} = -\frac{2x(1-x) - \alpha[x - (\gamma/\alpha)]^2}{x(1-x)[x - (\gamma/\alpha)]}$$

gives $w' = C_1 \mathrm{e}^{r(x)}$, where C_1 is an arbitrary constant and

$$r(x) = -\int \frac{2x(1-x) - \alpha[x - (\gamma/\alpha)]^2}{x(1-x)[x - (\gamma/\alpha)]}\, dx.$$

Evaluating the latter integral, one obtains

$$r(x) = \ln\Big(|x|^{-\gamma}\, |x-1|^{\gamma-\alpha}\, [x - (\gamma/\alpha)]^{-2}\Big),$$

and hence

$$w' = C_1\Big(x^{-\gamma}\, (x-1)^{\gamma-\alpha}\, [x - (\gamma/\alpha)]^{-2}\Big).$$

Thus, the solution of equation (4.40) is given by quadrature:

$$w = C_1 \int \Big(|x|^{-\gamma}\, |x-1|^{\gamma-\alpha}\, [x-(\gamma/\alpha)]^{-2}\Big)\, dx + C_2, \quad C_1, C_2 = \text{const.} \tag{4.41}$$

Substituting the expression (4.41) in the formula (4.39), we obtain two independent solutions of the original equation (4.34):

$$y_1(x) = \Big(x - \frac{\gamma}{\alpha}\Big) \int \Big(|x|^{-\gamma}\, |x-1|^{\gamma-\alpha}\, [x - (\gamma/\alpha)]^{-2}\Big)\, dx, \quad y_2(x) = x - \frac{\gamma}{\alpha}\cdot$$

Taking the linear combination of $y_1(x)$ and $y_2(x)$ with arbitrary constant coefficients we obtain the general solution (4.35) to Equation (4.34), thus completing the proof.

Remark 9.5. If γ and $\gamma - \alpha$ are rational numbers, one can reduce (4.41) to integration of a rational function by standard substitutions and represent the solution (4.41) in terms of elementary functions. See examples in Sections 4.3 and 6.

Corollary 9.1. Using the transformation (4.33) of equation (4.32) to the standard form (4.31), one can integrate Eq. (4.32) with $E = -\,C$:

$$(x^2 + Ax + B)\,y'' + (Cx + D)\,y' - C\,y = 0. \tag{4.42}$$

5 Integration of Eq. (1.1) using Lagrangian

Let us apply the integration method of Section 3.3 to Equation (1.1):

$$y'' - \frac{y'}{y^2} + \frac{1}{xy} = 0 \tag{5.1}$$

For this equation, we already know two symmetries (4.1),

$$X_1 = x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y}\,, \quad X_2 = 2x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\,, \tag{5.2}$$

and the invariant Lagrangian (4.30),

$$L = -\frac{1}{y^2} + \frac{y^2}{x^2} - 2\frac{yy'}{x} + y'^2 - 2\Big(\frac{1}{x} - \frac{y'}{y}\Big)\ln\Big|\frac{y^2}{x} - yy'\Big|. \tag{5.3}$$

Since the Lagrangian (5.3) has singularities, we will begin with singling out the associated singular solutions of Eq. (5.1).

5.1 Singularities of Lagrangian and singular solutions

Recall that the hypergeometric equation (4.26) has the singular points

$$z = 0 \qquad \text{and} \qquad z = 1. \tag{5.4}$$

According to definition of z given in (4.24), the singular points (5.4) provide two first-order differential equations,

$$y' = \frac{y}{x} \tag{5.5}$$

and

$$y' = \frac{y}{x} - \frac{1}{y}\,, \tag{5.6}$$

respectively. The Lagrangian (5.3) collapses at both singular points. Namely, L is not defined at the point $z = 0$, i.e. at (5.5), and vanishes identically at the point $z = 1$, i.e. at (5.6). See also Remark 9.4. Eq. (5.5) yields $y = Kx,\ K = \text{const.}$

Equation (5.6) can be integrated, e.g. by noting that it admits the operators (5.2). Indeed, equation (5.6) is identical with $z = 1$, where z is a differential invariant of X_1 and X_2. Hence, one can employ either *canonical variables* or *Lies's integrating factors.*

Canonical variables. Let us use, e.g. the operator X_1. The equations $X_1(\tau) = 1$ and $X_1(v) = 0$ provide the following canonical variables:

$$\tau = -\frac{1}{x}\,, \quad v = \frac{y}{x}\,\cdot$$

In this variables, equation (5.6) is written

$$\frac{dv}{d\tau} + \frac{1}{v} = 0$$

and yields

$$v = \mp\sqrt{-2t + C}.$$

Returning to the original variables, one obtains the following solution of equation (5.6):

$$y = \pm\sqrt{2x + Cx^2}\,. \tag{5.7}$$

One can also make use of the second symmetry (5.2). Then, solving the equations $X_2(\tau) = 1,\ X_2(v) = 0$ and assuming $x > 0$ for simplicity, one obtains

$$\tau = \ln x, \quad v = \frac{y}{\sqrt{x}}\,\cdot$$

In these canonical variables equation (5.6) becomes

$$\frac{dv}{d\tau} = \frac{v^2 - 2}{v}\,,$$

whence

$$v(\tau) = \pm\sqrt{2 + C\mathrm{e}^{2\tau}}\,\cdot$$

The substitution $y = \sqrt{x}\,v(\tau),\ \tau = \ln x$ yields the previous solution (5.7).

Integrating factors. Recall that a first-order differential equation

$$M(x,y)dx + N(x,y)dy = 0 \tag{5.8}$$

with a known infinitesimal symmetry

$$X = \xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y}$$

has the following integrating factor known as *Lie's integrating factor:*

$$\mu(x,y) = \frac{1}{\xi M + \eta N}\,\cdot \tag{5.9}$$

Furthermore, if one knows two linearly independent integrating factors $\mu_1(x,y)$ and $\mu_1(x,y)$, one can obtain the general solution of equation (5.8) from the algebraic relation

$$\frac{\mu_1(x,y)}{\mu_2(x,y)} = C \tag{5.10}$$

with an arbitrary constant C. Applying these two principles to equations (5.8) with two known infinitesimal symmetries, one obtains the general solution without integration.

Let us return to our equation (5.6). We rewrite it in the form (5.8):

$$(x - y^2)dx + xydy = 0$$

and use its two known infinitesimal symmetries (5.2):

$$X_1 = x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y}\,, \quad X_2 = 2x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\,\cdot$$

Lie's integrating factors (5.9) corresponding to X_1 and X_2, respectively, have the following forms:

$$\mu_1(x,y) = \frac{1}{x^2(x-y^2) + x^2y^2} = \frac{1}{x^3}\,,$$

$$\mu_2(x,y) = \frac{1}{2x(x-y^2) + xy^2} = \frac{1}{2x^2 - xy^2}\,\cdot$$

Therefore the algebraic relation (5.10) has the form

$$\frac{2x - y^2}{x^2} = C.$$

Upon solving it with respect to y, one obtains the singular solution (5.7).

We summarize: *The Lagrangian* (5.3) *has singularities only at the following singular solutions of equation* (5.1):

$$y = K\,x, \quad y = \pm\sqrt{2x + C\,x^2}\,, \qquad K,\, C = \text{const.} \tag{5.11}$$

Remark 9.6. The singular point $z = 1$ provides two singular solutions,

$$y = \sqrt{2x + C\,x^2}$$

and

$$y = -\sqrt{2x + C\,x^2},$$

because equation (5.1) is invariant under the reflection $y \to -y$ of the dependent variable.

5.2 General solution

Let us find now the regular solutions by means of the integration method discussed in Section 3.3. Since we already know the symmetries (5.2) and an invariant Lagrangian (5.3) for Eq. (5.1) we proceed to the *third step.*

Application of the formula (3.9) to the operators (5.2) yields:

$$T_1 = x^2\,L + x(y - xy')\,L_{y'}\,, \qquad T_2 = 2x\,L + (y - 2xy')\,L_{y'}\,.$$

Let us substitute here L and $L_{y'}$ from Eqs. (4.24) and (4.25), respectively:

$$L = \frac{1}{y^2}\Psi(z), \quad L_{y'} = -\frac{1}{y}\Psi'(z), \qquad z = \frac{y^2}{x} - yy',$$

where, according to (4.29),

$$\Psi(z) = z^2 - 1 - 2z\ln|z|, \quad \Psi'(z) = 2(z - 1 - \ln|z|).$$

Thus, we arrive at the following two linearly independent conserved quantities:

$$\begin{aligned} T_1 \ &= \frac{x^2}{y^2}\,\Psi(z) - \frac{x(y - xy')}{y}\,\Psi'(z) \equiv -\frac{x^2}{y^2}\,(1 - z)^2, \\ T_2 \ &= \frac{2x}{y^2}\,\Psi(z) - \frac{y - 2xy'}{y}\,\Psi'(z) \equiv -2\frac{x}{y^2}\,(1 - z)^2 - 2(z - 1 - \ln|z|). \end{aligned}$$

In the original variables x, y, and y' the conserved quantities are written

$$
\begin{aligned}
T_1 &= 2x - 2\frac{x^2y'}{y} - \frac{x^2}{y^2} - y^2 + 2xyy' - x^2y'^2\,,\\
T_2 &= 2\Big(1 - \frac{x}{y^2} - 2\frac{xy'}{y} + yy' - xy'^2 - \ln\Big|\frac{y^2}{x} - yy'\Big|\Big).
\end{aligned} \tag{5.12}
$$

However, the form (5.12) of the conserved quantities is not convenient for eliminating y' from the conservation laws. Therefore, we will use the following representation of T_1 and T_1 in terms of the differential invariant z :

$$
T_1 = -\frac{x^2}{y^2}(1-z)^2, \qquad T_1 - \frac{x}{2}\,T_2 = x(1 - z + \ln|z|). \tag{5.13}
$$

Then one can readily eliminate the variable z instead of y'.

Fourth step: Let us write the conservation laws (3.10) in the form

$$
T_1 = -\,C_1^2, \qquad T_2 = 2C_2.
$$

Using the expression of T_1 given in (5.13), we have

$$
\frac{x^2}{y^2}\,(1-z)^2 = C_1^2,
$$

whence

$$
1 - z = C_1\frac{y}{x}\,, \quad z = 1 - C_1\frac{y}{x}\cdot
$$

Substitution of the above expressions in the second equation (5.13) yields:

$$
-\,C_1^2 - C_2x = x\Big(C_1\,\frac{y}{x} + \ln\Big|C_1\,\frac{y}{x} - 1\Big|\Big).
$$

Hence, the two-parameter solution (3.11) is given in the implicit form:

$$
C_1y + C_2x + C_1^2 + x\ln\Big|C_1\,\frac{y}{x} - 1\Big| = 0.
$$

Invoking two singular solutions (5.11), we see that the complete set of solutions to equation (5.1) is given by the following *distinctly different formulae* (cf. [39]):

$$
\begin{aligned}
&y = K\,x, \qquad y = \pm\sqrt{2x + C\,x^2}\,,\\
&C_1y + C_2x + C_1^2 + x\ln\Big|C_1\,\frac{y}{x} - 1\Big| = 0.
\end{aligned} \tag{5.14}
$$

Remark 9.7. The representation of the solution by the different formulae (5.14) does not conflict with uniqeness of the solution to the Cauchy problem. Indeed, any initial data $x = x_0,\ y(x_0) = y_0,\ y'(x_0) = y'_0$ is compatible only with one formula (5.14) chosen in accordance with the initial value $z_0 = (y_0^2/x_0) - y_0 y'_0$ of the invariant z. Namely, the solution with the initial data $x = x_0,\ y(x_0) = y_0,\ y'(x_0) = y'_0$ is given by the first or the second formula (5.14) if $z_0 = 0$ or $z_0 = 1$, respectively. Otherwise it is given by the third formula (5.14). The constants K, C, C_1, C_2 are found by substituting $x = x_0,\ y = y_0,\ y' = y_0$ in the formulae (5.14) together with their differential consequences:

$$
\begin{aligned}
& y' = K, \qquad y' = \pm\frac{1 + Cx}{\sqrt{2x + C\,x^2}}\,, \\
& C_1\,y' + C_2 + \ln\left|C_1\,\frac{y}{x} - 1\right| + \frac{C_1(xy' - y)}{C_1\,y - x} = 0,
\end{aligned}
\tag{5.15}
$$

Examples of initial value problems.

(i) Let $x_0 = 1,\ y_0 = 1,\ y'_0 = 1$. Then $z_0 = 0$, and hence the solution belongs to the first formula (5.14). The substitution $x = 1,\ y = 1,\ y' = 1$ in (5.14) and (5.15) yields $K = 1$. Hence, the solution of equation (5.1) with the initial data $x_0 = 1,\ y_0 = 1,\ y'_0 = 1$ has the form $y = x$.

(ii) For the initial data $x_0 = 1,\ y_0 = 1,\ y'_0 = 2$, one has $z_0 = -1$. Therefore the solution belongs to the third formula (5.14). Substituting the initial values of x, y, y' (5.14)-(5.15), one obtains $C_1 = 2,\ C_2 = -6$, and hence $2y - 6x + 4 + x\ln|(2y/x) - 1| = 0$.

(iii) For the initial data $x_0 = 1,\ y_0 = 1,\ y'_0 = -1$, one has $z_0 = 2$. Accordingly, the solution is given by the third formula (5.14) with $C_1 = -1,\ C_2 = -\ln 2$.

(iv) If $x_0 = 1,\ y_0 = 1,\ y'_0 = 0$, one has $z_0 = 1$. Hence, we should use the second formula (5.14), where we have to specify the sign and determine the constat C by substituting the initial values $x_0 = 1,\ y_0 = 1,\ y'_0 = 0$ in (5.14)-(5.15). The reckoning shows that the solution is given by the second formula (5.14) with the positive sign and $C = -1$, i.e. it has the form $y = \sqrt{2x - x^2}$.

(v) Likewise, taking $x_0 = 1,\ y_0 = -1,\ y'_0 = 0$, one can verify that the solution of the Cauchy problems is given by the second formula (5.14) with the negative sign and $C = -1$, i.e. $y = -\sqrt{2x - x^2}$.

6 Application of the method to Eq. (1.2)

Consider now Equation (1.2):

$$y'' = \mathrm{e}^y - \frac{y'}{x}\,\cdot$$

It has the following two symmetries ([39], Section 9.3.1):

$$X_1 = x\ln|x|\frac{\partial}{\partial x} - 2(1+\ln|x|)\frac{\partial}{\partial y}\,, \quad X_2 = x\frac{\partial}{\partial x} - 2\frac{\partial}{\partial y}\,\cdot \tag{6.1}$$

The determining equation (3.4) for the Lagrangians of equation (1.2) has the form

$$\left(\mathrm{e}^y - \frac{y'}{x}\right) L_{y'y'} + y' L_{yy'} + L_{xy'} - L_y = 0, \qquad L_{y'y'} \neq 0. \tag{6.2}$$

6.1 Calculation of the invariant Lagrangian

Let us find the invariant Lagrangian for our equation using both symmetries (6.1). The invariance test (3.5) under the operators (6.1) yields:

$$\begin{aligned} & x\ln|x|\frac{\partial L}{\partial x} - 2(1+\ln|x|)\frac{\partial L}{\partial y} \\ & - \left[\frac{2}{x} + (1+\ln|x|)\,y'\right]\frac{\partial L}{\partial y'} + (1+\ln|x|)L = 0, \\ & x\frac{\partial L}{\partial x} - 2\frac{\partial L}{\partial y} - y'\frac{\partial L}{\partial y'} + L = 0. \end{aligned} \tag{6.3}$$

Looking for the solution of the system (6.3) in the implicit form

$$V(x, y, y', L) = 0$$

one arrives at the system of two homogeneous equations

$$Z_1(V) = 0, \qquad Z_2(V) = 0 \tag{6.4}$$

with the following linear differential operators:

$$
\begin{aligned}
Z_1 &= x\ln|x|\frac{\partial}{\partial x} - 2(1+\ln|x|)\frac{\partial}{\partial y}\\
&\quad - \Big(\frac{2}{x} + (1+\ln|x|)y'\Big)\frac{\partial}{\partial y'} - (1+\ln|x|)L\frac{\partial}{\partial L}\,,\\
Z_2 &= x\frac{\partial}{\partial x} - 2\frac{\partial}{\partial y} - y'\frac{\partial}{\partial y'} - L\frac{\partial}{\partial L}\,\cdot
\end{aligned}
$$

The invariants for Z_1 are

$$\lambda = x\,\mathrm{e}^{y/2}\ln|x|, \quad \mu = (2+xy')\ln|x|, \quad \nu = L\,\mathrm{e}^{-y/2}.$$

Hence, the first equation $Z_1(V) = 0$ yields that

$$V = V(\lambda, \mu, \nu).$$

Furthermore, we have

$$Z_2(\lambda) = x\,\mathrm{e}^{y/2} = \frac{\lambda}{\ln|x|}\,, \quad Z_2(\mu) = 2 + xy' = \frac{\mu}{\ln|x|}\,, \quad Z_2(\nu) = 0.$$

Thus, we have for $V = V(\lambda, \mu, \nu)$:

$$Z_2(V) = \frac{1}{\ln|x|}\left(\lambda\frac{\partial V}{\partial\lambda} + \mu\frac{\partial V}{\partial\mu}\right).$$

Hence, the second equation (6.4) is identical with the equation $\widetilde{Z}_2(V) = 0$, where

$$\widetilde{Z}_2 = \lambda\frac{\partial}{\partial\lambda} + \mu\frac{\partial}{\partial\mu}\,\cdot$$

Solutions $z = \mu/\lambda$ and ν to the characteristic equation

$$\frac{d\lambda}{\lambda} = \frac{d\mu}{\mu}$$

provide the following common invariants of Z_1 and Z_2 :

$$z = \left(\frac{2}{x} + y'\right)\mathrm{e}^{-y/2}\,, \quad \nu = L\,\mathrm{e}^{-y/2}.$$

Thus, $V = V(z,\nu)$. The equation $V(z,\nu) = 0$ when solved for ν yields $\nu = \Psi(x)$, or:

$$L = \mathrm{e}^{y/2}\,\Psi(z), \quad z = \left(\frac{2}{x} + y'\right)\mathrm{e}^{-y/2}\,. \tag{6.5}$$

We have:

$$z_x = -\frac{2}{x^2}\mathrm{e}^{-y/2}\,, \quad z_y = -\frac{z}{2}\,, \quad z_{y'} = \mathrm{e}^{-y/2}\,,$$

and therefore

$$L_{y'} = \Psi'\,, \quad L_{y'y'} = \mathrm{e}^{-y/2}\,\Psi''\,, \quad L_{yy'} = -\frac{z}{2}\Psi''\,,$$

$$L_{xy'} = -\frac{2}{x^2}\,\mathrm{e}^{-y/2}\,\Psi''\,, \quad L_y = \frac{1}{2}\,\mathrm{e}^{y/2}\left(\Psi - z\,\Psi'\right). \tag{6.6}$$

Substitution of these expressions reduces the equation (6.2) to the integrable linear ordinary differential equation of the form (4.42) with the coefficients $C = -1$, $B = -2$ and $A = D = 0$:

$$(z^2 - 2)\Psi'' - z\Psi' + \Psi = 0. \tag{6.7}$$

In accordance with Section 4.4, we rewrite equation (6.7) in the new independent variable t defined by (4.33), where we replace x and x_1, x_2 by z and $z_1 = \sqrt{2}$, $z_2 = -\sqrt{2}$, respectively. Thus, we let

$$z = \sqrt{2}\,(1 - 2\,t) \tag{6.8}$$

and arrive at the hypergeometric equation of the form (4.34) with $\alpha = -1$ and $\gamma = -1/2$:

$$t(1-t)\Psi'' + \left(t - \frac{1}{2}\right)\Psi' - \Psi = 0, \tag{6.9}$$

where Ψ' denotes the differentiation with respect to t. The solution to equation (6.9) is given by the formula (4.35) and has the form

$$\Psi(t) = (2t - 1)(M\mathcal{J} + N), \tag{6.10}$$

where M, N are arbitrary constants and $\mathcal{J}$ is the following integral:

$$\mathcal{J} = 4\int \frac{\sqrt{|t(t-1)|}}{(2t-1)^2}\,dt. \tag{6.11}$$

Let us evaluate the integral $\mathcal{J}$ and express the solution (6.10) in elementary functions.

We will first assume that $t(t-1) > 0$, i.e. either $t > 1$ or $t < 0$. According to (6.8), it means that

$$z^2 - 2 > 0. \tag{6.12}$$

Using this assumption, let us rewrite the integral $\mathcal{J}$ in the form

$$\mathcal{J} = 4\int \frac{t\sqrt{(t-1)/t}}{(2t-1)^2}\,dt. \tag{6.13}$$

The standard substitution $(t-1)/t = s^2$ together with the expressions

$$\sqrt{\frac{t-1}{t}} = s, \quad t = \frac{1}{1-s^2}\,, \quad dt = \frac{2sds}{(1-s^2)^2}\,, \quad 2t-1 = \frac{1+s^2}{1-s^2}\,, \tag{6.14}$$

transforms the integral (6.11) to the form

$$\mathcal{J} = 8\int \frac{s^2 ds}{(1-s^2)(1+s^2)^2} = \int \left(\frac{1}{s+1} - \frac{1}{s-1} + \frac{2}{s^2+1} - \frac{4}{(s^2+1)^2}\right) ds\,.$$

Since

$$\int \frac{ds}{(s^2+1)^2} = \frac{s}{2(s^2+1)} + \frac{1}{2}\int \frac{ds}{s^2+1}\,,$$

the integral $\mathcal{J}$ reduces to

$$\mathcal{J} = \ln\left|\frac{s+1}{s-1}\right| - \frac{2s}{s^2+1}\,\cdot$$

Invoking the definition (6.14) of s and using the assumption (6.12), one obtains the following expression for the integral (6.11):

$$\mathcal{J} = \ln\left|\frac{\sqrt{|t|}+\sqrt{|t-1|}}{\sqrt{|t|}-\sqrt{|t-1|}}\right| - 2\frac{\sqrt{t(t-1)}}{2t-1} \equiv \ln\left[\sqrt{|t|}+\sqrt{|t-1|}\right]^2 - 2\frac{\sqrt{t(t-1)}}{2t-1}\,\cdot$$

Substituting $\mathcal{J}$ in (6.10) one obtains the following solution of Eq. (6.9):

$$\Psi(t) = M\left[(2t-1)\,\ln\left(|t| + |t-1| + 2\sqrt{t(t-1)}\right) - 2\sqrt{t(t-1)}\right] + (2t-1)N.$$

Using the definition (6.8) of z, the inequality (6.12) and the equations

$$2t-1 = -\frac{z}{\sqrt{2}}\,, \quad |t| + |t-1| = \frac{|z|}{\sqrt{2}}\,, \quad 2\sqrt{t(t-1)} = \frac{\sqrt{z^2-2}}{\sqrt{2}}\,,$$

we have

$$\Psi(z) = -\frac{M}{\sqrt{2}}\left[\sqrt{z^2-2} + z\ln(\sqrt{z^2-2}+|z|)\right] + \frac{M\ln\sqrt{2}-N}{\sqrt{2}}\, z\,.$$

We simplify the above expression by taking $M = -\sqrt{2}$, $N = -\sqrt{2}\ln\sqrt{2}$ and obtain

$$\Psi(z) = \sqrt{z^2-2} + z\ln(\sqrt{z^2-2}+|z|). \tag{6.15}$$

Finally, substituing (6.15) in (6.5), we arrive at the following Lagrangian:

$$L = \mathcal{B} + \left(\frac{2}{x}+y'\right)\left\{\ln\left(\mathcal{B}+\left|\frac{2}{x}+y'\right|\right) - \frac{y}{2}\right\}, \tag{6.16}$$

where

$$\mathcal{B} = \sqrt{\left(\frac{2}{x}+y'\right)^2 - 2\mathrm{e}^y}\,.$$

Using (6.6) and (6.15) one obtains:

$$L = \mathcal{B} + \left(\frac{2}{x}+y'\right)\left\{\ln\left(\mathcal{B}+\left|\frac{2}{x}+y'\right|\right) - \frac{y}{2}\right\},$$

$$L_y = \frac{1}{2}\mathcal{B},\ \ L_{y'} = -\frac{y}{2} + \ln\left(\mathcal{B}+\left|\frac{2}{x}+y'\right|\right),$$

$$L_{y'y'} = -\frac{1}{\mathcal{B}}, \quad L_{yy'} = \frac{1}{2\mathcal{B}}\left(\frac{2}{x}+y'\right), \quad L_{xy'} = \frac{2}{x^2\mathcal{B}}\,\cdot \tag{6.17}$$

Thus, we have for the Lagrangian (6.16):

$$\frac{\delta L}{\delta y} = \frac{1}{\mathcal{B}}\left(y'' + \frac{y'}{x} - \mathrm{e}^y\right).$$

Remark 9.8. By reducing $\Psi(z)$ to (6.15) we eliminated the regular solution $\Psi = z$ of equation (6.7). The reason for the elimination is that $\Psi = z$ leads to the trivial Lagrangian (6.5), $L^* = y' + (2/x)$.

6.2 Singularities of Lagrangian and singular solutions

According to (6.8), the singular points $t = 0$ and $t = 1$ of the hypergeometric equation (6.9) correspond to $z = \sqrt{2}$ and $z = -\sqrt{2}$, respectively. Hence, by definition (6.5) of z, the singularities of the Lagrangian furnish two first-order equations:

$$\frac{dy}{dx} + \frac{y}{x} = \pm\sqrt{2}\,\mathrm{e}^{y/2} \tag{6.18}$$

differing from each other merely by the sign in the right-hand side. Let us integrate equation (6.18) by the methods described in Section 5.1.

Let us employ the method of canonical variables by taking the second operator from (6.1):

$$X_2 = x\frac{\partial}{\partial x} - 2\frac{\partial}{\partial y}\cdot$$

Solving the equations $X_2(\tau) = 1,\ X_2(v) = 0$, one obtains

$$\tau = \ln|x|, \quad v = y + \ln(x^2)\,.$$

In the canonical variables equation (6.18) becomes

$$\frac{dv}{d\tau} = \pm\sqrt{2}\,\mathrm{e}^{v/2}\,.$$

The integration yields

$$v(\tau) = \ln\left[\frac{2}{(C \mp \tau)^2}\right]\cdot$$

Substituting

$$y = v(\tau) - \ln(x^2), \quad \tau = \ln|x|,$$

one obtains the solution to equation (6.18):

$$y = \ln\left[\frac{2}{(Cx \mp x\ln|x|)^2}\right]. \tag{6.19}$$

Let us solve Equation (6.18) by the method of Lie's integrating factors using both symmetries (6.1). Rewriting Equation (6.18) in the form (5.8):

$$\left(2 \mp \sqrt{2}\,x\,\mathrm{e}^{y/2}\right)dx + xdy = 0$$

and applying the formula (5.9) to the symmetries (5.2), one obtains two integrating factors:

$$\mu_1(x,y) = \frac{1}{\mp\sqrt{2}\,x^2\,\mathrm{e}^{y/2}\,\ln|x| - 2x}\,, \quad \mu_2(x,y) = \frac{1}{\mp\sqrt{2}\,x^2\,\mathrm{e}^{y/2}}\cdot$$

The algebraic relation $\mu_2/\mu_1 = C$ (cf. Equation (5.10)) has the form

$$\ln|x| \pm \frac{\sqrt{2}}{x}\,\mathrm{e}^{y/2} = C.$$

Upon solving it with respect to y, one obtains the solution (6.19).

6.3 General solution

In this example, the procedure for constructing the regular solutions is similar to that given in detail in Section 5.2.

Application of the formula (3.9) to the operators (6.1) yields:

$$T_1 = (x\ln|x|)\,L - (2 + 2\ln|x| + y'x\ln|x|)\,L_{y'}\,, \qquad T_2 = x\,L - (2 + xy')\,L_{y'}\,.$$

Let us substitute here L and $L_{y'}$ from equations (6.5) and (6.6), respectively:

$$L = \mathrm{e}^{y/2}\Psi(z), \quad L_{y'} = \Psi'(z), \qquad z = \left(\frac{2}{x} + y'\right)\mathrm{e}^{-y/2}\,,$$

where, according to (6.15),

$$\Psi(z) = \sqrt{z^2-2} + z\ln(\sqrt{z^2-2} + |z|), \quad \Psi'(z) = \ln(\sqrt{z^2-2} + |z|).$$

The reckoning shows that

$$T_1 = (x\ln|x|)\,\mathrm{e}^{y/2}\sqrt{z^2-2} - 2\ln(\sqrt{z^2-2} + |z|), \quad T_2 = x\,\mathrm{e}^{y/2}\sqrt{z^2-2}. \tag{6.20}$$

Let us begin the fourth step of the integration procedure with the simplest conservation law $T_2 = C_2$:

$$x\,\mathrm{e}^{y/2}\sqrt{z^2-2} = C_2.$$

It follows:

$$z^2 - 2 = C_2^2 x^{-2}\mathrm{e}^{-y}\,, \quad |z| = \frac{\mathrm{e}^{-y/2}}{|x|}\sqrt{C_2^2 + 2x^2\mathrm{e}^y}\,.$$

Substitution of the above expressions in the conservation law $T_1 = K_1$ and simple calculations yield:

$$C_2\ln|x| - 2\ln\left(\frac{\mathrm{e}^{-y/2}}{|x|}\right) - 2\ln\left|C_2 + \sqrt{C_2^2 + 2x^2\mathrm{e}^y}\right| = K_1,$$

or

$$\ln\left[|x|^{-C_2/2}\left(C_2\,\frac{\mathrm{e}^{-y/2}}{|x|} + \sqrt{C_2^2 x^{-2}\mathrm{e}^{-y} + 2}\right)\right] = -\frac{K_1}{2}.$$

It follows:

$$\sqrt{C_2^2 x^{-2}\mathrm{e}^{-y} + 2} = C_1\,|x|^{C_2/2} - C_2\,|x|^{-1}\,\mathrm{e}^{-y/2},$$

where $C_1 = \mathrm{e}^{-K_1/2} > 0$. Thus

$$\mathrm{e}^y = 4C_1^2 C_2^2 \frac{|x|^{C_2-2}}{(C_1^2|x|^{C_2} - 2)^2}\,,$$

and hance the two-parameter solution (3.11) has the form

$$y = \ln\left[\frac{2C_1C_2}{C_1^2|x|^{C_2} - 2}\right]^2 + (C_2 - 2)\ln|x|. \tag{6.21}$$

We will consider now the case $t(t-1) < 0$, i.e. $0 < t < 1$. In other words, we let

$$z^2 - 2 < 0. \tag{6.22}$$

Then the integral $\mathcal{J}$ has the form

$$\mathcal{J} = 4\int \frac{\sqrt{t(t-1)}}{(2t-1)^2}\,dt. \tag{6.23}$$

The substitution

$$s = \sqrt{\frac{1-t}{t}}\,, \quad t = \frac{1}{1+s^2}\,, \quad dt = -\frac{2sds}{(1+s^2)^2}$$

maps it to

$$\mathcal{J} = -8\int \frac{s^2 ds}{(1+s^2)(1-s^2)^2} = \int\left(\frac{2}{1+s^2} - \frac{1}{(s-1)^2} - \frac{1}{(s+1)^2}\right)ds\,.$$

Hence

$$\mathcal{J} = \frac{2s}{s^2-1} + 2\arctan s = 2\arctan\sqrt{\frac{1-t}{t}} - 2\frac{\sqrt{t(1-t)}}{2t-1}\,\cdot$$

Following the reasoning used previously in the assumption (6.12), we take the solution of equation (6.9) in the form

$$\Psi(t) = \frac{1}{2}(2t-1)\mathcal{J} = (2t-1)\arctan\sqrt{\frac{1-t}{t}} - \sqrt{t(1-t)}$$

and ultimately obtain

$$\Psi(z) = \sqrt{2-z^2} + 2z\,\arctan\sqrt{\frac{\sqrt{2}+z}{\sqrt{2}-z}}\,\cdot \tag{6.24}$$

instead of (6.15).

Thus, we have:

$$L = \mathrm{e}^{y/2}\Psi(z), \quad L_{y'} = \Psi'(z), \qquad z = \left(\frac{2}{x} + y'\right)\mathrm{e}^{-y/2},$$

where

$$\Psi(z) = \sqrt{2 - z^2} + 2z \ \arctan\sqrt{\frac{\sqrt{2} + z}{\sqrt{2} - z}}, \quad \Psi'(z) = 2 \ \arctan\sqrt{\frac{\sqrt{2} + z}{\sqrt{2} - z}} \cdot$$

Substituting the above expressions for L and $L_{y'}$ in the expressions

$$T_1 = (x \ln|x|)\, L - (2 + 2\ln|x| + y'x\ln|x|)\, L_{y'}, \quad T_2 = x\, L - (2 + xy')\, L_{y'},$$

obtained by applying the formula (3.9) to the operators (6.1), we arrive at the following conserved quantities:

$$T_1 = (x \ln x)\, \mathrm{e}^{y/2}\sqrt{2 - z^2} - 4 \arctan\sqrt{\frac{\sqrt{2} + z}{\sqrt{2} - z}}, \quad T_2 = x\, \mathrm{e}^{y/2}\sqrt{2 - z^2}.$$

The conservation laws (3.10) written in the form

$$T_1 = -2K_1, \quad T_2 = 2K_2$$

yield:

$$x\, \mathrm{e}^{y/2}\sqrt{2 - z^2} = 2K_2, \quad K_2 \ln|x| - 2 \arctan\sqrt{\frac{\sqrt{2} + z}{\sqrt{2} - z}} = -K_1.$$

One can readily eliminate z from the conservation laws by solving each of the above equations with respect to z^2. The first equation yields:

$$z^2 = 2\left(1 - 2K_2^2\, x^{-2}\, \mathrm{e}^{-y}\right).$$

Using the notation

$$\theta = (K_1/2) + (K_2/2)\ln|x|$$

one obtains from the second equation:

$$z^2 = 2\left(\frac{1 - \tan^2\theta}{1 - \tan^2\theta}\right)^2 \equiv 2\cos^2(2\theta).$$

Consequently,

$$1 - 2K_2^2\, x^{-2}\, \mathrm{e}^{-y} = \cos^2(2\theta).$$

Hence, substituting the expression for θ, one has:

$$2K_2^2\, x^{-2}\, \mathrm{e}^{-y} = \sin^2(K_1 + K_2\, \ln|x|).$$

Thus, we have the following solution of equation (1.2):

$$y = \ln\left[\frac{2K_2^2}{x^2 \sin^2(K_1 + K_2\, \ln|x|)}\right] \cdot \tag{6.25}$$

The singular solutions (6.19) together with (6.21) and (6.25) furnish the complete set of solutions equation of (1.2).

Thus, the general solution to Equation (1.2),

$$y'' = \mathrm{e}^y - \frac{y'}{x}\,,$$

is given by the following three *distinctly different formulae*:

$$\begin{aligned} y &= \ln\left[\frac{2}{(Cx \pm x\ln|x|)^2}\right], \\ y &= \ln\left[\frac{2C_1C_2}{C_1^2|x|^{C_2} - 2}\right]^2 + (C_2 - 2)\ln|x|, \\ y &= \ln\left[\frac{2K_2^2}{x^2\sin^2(K_1 + K_2\, \ln|x|)}\right]. \end{aligned} \tag{6.26}$$

Paper 10

Formal Lagrangians

Unabridged Preprint [45]. See also [46], [50]

Abstract. A general theorem on derivation of conservation laws from symmetries is proved for arbitrary differential equations and systems of differential equations where the number of equations is equal to the number of dependent variables. The new conservation theorem is based on a concept of a formal Lagrangian and adjoint equations for non-linear equations.

I dedicate this paper to L.V. Ovsyannikov
on the special occasion of his 88th birthday.

In 1973, during a discussion of contemporary works in soliton theory at "Theoretical seminar of the Institute of Hydrodynamics" in Novosibirsk, Professor Lev V. Ovsyannikov asked me if the infinite number of conservation laws for the Korteweg-de Vries (KdV) equation could be obtained from its symmetries. The answer was by no means evident because the KdV equation did not have the usual Lagrangian, and hence the Noether theorem was not applicable. In the present paper I give the affirmative answer to Ovsyannikov's question by using my recent result on conservation laws applicable to arbitrary differential equations. The new theorem, *Theorem on nonlocal conservation laws*, is based on the fact that *the adjoint equations inherit all Lie point and contact, Lie-Bäcklund and non-local symmetries of the original equations.* For derivation of the infinite series of conservation laws for the KdV equation, I modify the notion of self-adjoint equations and extend it to non-linear equations.

1 Introduction

Recall the formulation of the well-known conservation theorem proved by Noether [87] in 1918 by using the calculus of variations. Let us consider variational integrals

$$\int_V \mathcal{L}(x, u, u_{(1)}, \ldots, u_{(s)})dx, \tag{1.1}$$

where $\mathcal{L}(x, u, u_{(1)}, \ldots, u_{(s)})$ is an sth-order Lagrangian, i.e. it involves, along with the independent variables $x = (x^1, \ldots, x^n)$ and the dependent variables $u = (u, \ldots, u^m)$, the first-order derivatives $u_{(1)} = \{u_i^\alpha\}$ of u with respect to x, i.e. $u_i^\alpha = D_i(u^\alpha)$, the second-order derivatives $u_{(2)} = \{u_{ij}^\alpha\}$, etc. up to derivatives $u_{(s)}$ of order s.

We will discuss here Noether's theorem in the case of Lagrangians up to the third order, i.e. when $s \leq 3$.

Theorem 10.1. Let the variational integral (1.1) with a first-order Lagrangian $\mathcal{L}(x, u, u_{(1)})$ be invariant under a group G with a generator

$$X = \xi^i(x, u, u_{(1)}, \ldots)\frac{\partial}{\partial x^i} + \eta^\alpha(x, u, u_{(1)}, \ldots)\frac{\partial}{\partial u^\alpha}\,\cdot \tag{1.2}$$

Then the vector field $C = (C^1, \ldots, C^n)$ defined by

$$C^i = \xi^i \mathcal{L} + (\eta^\alpha - \xi^j u_j^\alpha)\frac{\partial \mathcal{L}}{\partial u_i^\alpha}\,, \quad i = 1, \ldots, n, \tag{1.3}$$

provides a conservation law for the Euler-Lagrange equations

$$\frac{\delta \mathcal{L}}{\delta u^\alpha} \equiv \frac{\partial \mathcal{L}}{\partial u^\alpha} - D_i\left(\frac{\partial \mathcal{L}}{\partial u_i^\alpha}\right) = 0, \quad \alpha = 1, \ldots, m. \tag{1.4}$$

In other words, the vector field (1.3) obeys the equation $\text{div}C \equiv D_i(C^i) = 0$ for all solutions of Eqs. (1.4), i.e.

$$D_i(C^i)\Big|_{(1.4)} = 0. \tag{1.5}$$

Any vector field C^i satisfying (1.5) is called a *conserved vector* for Eqs. (1.4).

In the case of second-order Lagrangians $L(x, u, u_{(1)}, u_{(2)})$ the Euler-Lagrange equations (1.4) and the conserved vector (1.3) are replaced by

$$\frac{\delta \mathcal{L}}{\delta u^\alpha} \equiv \frac{\partial \mathcal{L}}{\partial u^\alpha} - D_i\left(\frac{\partial \mathcal{L}}{\partial u_i^\alpha}\right) + D_i D_k\left(\frac{\partial \mathcal{L}}{\partial u_{ik}^\alpha}\right) = 0 \tag{1.6}$$

and

$$C^i = \xi^i \mathcal{L} + (\eta^\alpha - \xi^j u_j^\alpha) \left[\frac{\partial \mathcal{L}}{\partial u_i^\alpha} - D_k \Big(\frac{\partial \mathcal{L}}{\partial u_{ik}^\alpha} \Big) \right] + D_k \big(\eta^\alpha - \xi^j u_j^\alpha \big) \frac{\partial \mathcal{L}}{\partial u_{ik}^\alpha} \, , \quad (1.7)$$

respectively.

In the case of third-order Lagrangians $L(x, u, u_{(1)}, u_{(2)}, u_{(3)})$ the Euler-Lagrange equations are written:

$$\frac{\delta \mathcal{L}}{\delta u^\alpha} \equiv \frac{\partial \mathcal{L}}{\partial u^\alpha} - D_i \left(\frac{\partial \mathcal{L}}{\partial u_i^\alpha} \right) + D_i D_k \left(\frac{\partial \mathcal{L}}{\partial u_{ik}^\alpha} \right) - D_i D_j D_k \left(\frac{\partial \mathcal{L}}{\partial u_{ijk}^\alpha} \right) = 0 \quad (1.8)$$

and the conserved vector (1.3) is replaced by

$$C^i = \xi^i \mathcal{L} + (\eta^\alpha - \xi^j u_j^\alpha) \left[\frac{\partial \mathcal{L}}{\partial u_i^\alpha} - D_j \Big(\frac{\partial \mathcal{L}}{\partial u_{ij}^\alpha} \Big) + D_j D_k \Big(\frac{\partial \mathcal{L}}{\partial u_{ijk}^\alpha} \Big) \right] \quad (1.9)$$

$$+ D_j \big(\eta^\alpha - \xi^j u_j^\alpha \big) \left[\frac{\partial \mathcal{L}}{\partial u_{ij}^\alpha} - D_k \Big(\frac{\partial \mathcal{L}}{\partial u_{ijk}^\alpha} \Big) \right] + D_j D_k \big(\eta^\alpha - \xi^j u_j^\alpha \big) \frac{\partial \mathcal{L}}{\partial u_{ijk}^\alpha} \cdot$$

Remark 10.1. It is manifest from Eq. (1.5) that any linear combination of conserved vectors is a conserved vector. Furthermore, any vector vanishing on the solutions of Eqs. (1.4) is a conserved vector, a *trivial conserved vector,* for Eqs. (1.4). In what follows, conserved vectors will be considered up to addition of trivial conserved vectors.

The invariance of the integral (1.1) implies that the Euler-Lagrange equations (1.4) admit the group G. Therefore, in order to apply Noether's theorem, one has first of all to find the symmetries of Eqs. (1.4). Then one should single out the symmetries leaving invariant the variational integral (1.4). This can be done by means of the following infinitesimal test for the invariance of the variational integral (see [34] or [39]):

$$X(\mathcal{L}) + \mathcal{L} D_i(\xi^i) = 0, \quad (1.10)$$

where the generator X is prolonged to the first derivatives $u_{(1)}$ by the formula

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha} + \Big[D_i(\eta^\alpha) - u_j^\alpha D_i(\xi^j) \Big] \frac{\partial}{\partial u_i^\alpha} \cdot \quad (1.11)$$

If Eq. (1.10) is satisfied, then the vector (1.3) provides a conservation law.

The invariance of the variational integral is sufficient, as said above, for the invariance of the Euler-Lagrange equations, but not necessary. Indeed, the following lemma shows that if one adds to a Lagrangian the divergence of any vector field, the Euler-Lagrange equations remain invariant.

Lemma 10.1. A function $f(x, u, \ldots, u_{(s)}) \in \mathcal{A}$ with several independent variables $x = (x^1, \ldots, x^n)$ and several dependent variables $u = (u^1, \ldots, u^m)$ is the divergence of a vector field $H = (h^1, \ldots, h^n),\ h^i \in \mathcal{A}$, i.e.

$$f = \mathrm{div} H \equiv D_i(h^i), \tag{1.12}$$

if and only if the following equations hold identically in $x, u, u_{(1)}, \ldots$:

$$\frac{\delta f}{\delta u^\alpha} = 0, \quad \alpha = 1, \ldots, m. \tag{1.13}$$

Therefore, one can add to the Lagrangian $\mathcal{L}$ the divergence of an arbitrary vector field depending on the group parameter and replace the invariance condition (1.10) by the divergence condition

$$X(\mathcal{L}) + \mathcal{L} D_i(\xi^i) = D_i(B^i). \tag{1.14}$$

Then Eqs. (1.4) are again invariant and have a conservation law $D_i\big(C^i\big) = 0$, where (1.3) is replaced by (see also Bessel-Hagen's paper in this volume)

$$C^i = \xi^i \mathcal{L} + (\eta^\alpha - \xi^j u_j^\alpha) \frac{\partial \mathcal{L}}{\partial u_i^\alpha} - B^i. \tag{1.15}$$

The restriction to Euler-Lagrange equations reduces applications of the Noether theorem significantly. For example, Noether's theorem is not applicable to *all evolution type equations*, to differential equations of an *odd order,* etc. Moreover, a symmetry of Euler-Lagrange equations should satisfy an additional property to leave invariant the variational integral. In spite of the fact that certain attempts have been made to overcome these restrictions and various generalizations of Noether's theorem have been discussed, I do not know in the literature a general result associating a conservation law with *every infinitesimal symmetry of an arbitrary differential equation.*

I present here a general result (Theorem 10.7) applicable, unlike Noether's theorem, to arbitrary differential equations (see also [46]). The new theorem is based on a concept of adjoint equations for non-linear equations and does not require existence of Lagrangians. The crucial fact is that all Lie point, Lie-Bäcklund and nonlocal symmetries of any equation are inherited by the adjoint equation (Section 3.1, Theorem 10.4). I give in Section 3.2 an explicit formula for the conserved quantities associated with these symmetries. Accordingly, one can find for any differential equation with known Lie, Lie-Bäcklund or nonlocal symmetries the associated conservation laws independently on existence of classical Lagrangians. The theorem is valid also for any system of differential equations where the number of equations is equal to the number of dependent variables (Theorem 10.5).

The new conservation theorem is, in fact, a *theorem on nonlocal conservation laws,* since the conserved quantities provided by this theorem are essentially "nonlocal". Namely, they involve, along with the variables of the equations under consideration, also *adjoint variables* which can be treated as *nonlocal variables* as defined, e.g. in [1]. In order to single out *local conservation laws,* I modify the notion of "self-adjoint" equations and show that the adjoint variables can be eliminated from the conserved quantities for the self-adjoint equations. However, this elimination may reduce some of nonlocal conservation laws to trivial local conservation laws. For example, the KdV and modified KdV equations are self-adjoint. It is shown in Section 4 that the known infinite series of *local conservation laws* of the KdV equation are associated with its *nonlocal symmetries.* On the other hand, the *local symmetries* of the KdV equation lead to essentially *nonlocal conservation laws* which become trivial if one eliminates the adjoint variable.

Finally, it is shown in Section 6.1 that even for equations having Lagrangians my theorem leads to conservation laws different from those given by Noether's theorem. This happens, for example in the case of nonlinear equations that are not self-adjoint.

2 Preliminaries

Let us begin with a brief discussion of the space $\mathcal{A}$ of differential functions, the basic operators $X, \delta/\delta u^\alpha, \mathcal{N}^i$ acting in $\mathcal{A}$ and the *fundamental identity* connecting them (see [34], Chap. 4 and 5; also [39], Sections 8.4 and 9.7). Then I recall the definition of adjoint equations to arbitrary equations and a new concept of self-adjoint equations [46].

2.1 Notation

Let $x = (x^1, \ldots, x^n)$ be n independent variables, and $u = (u^1, \ldots, u^m)$ be m dependent variables. We will use the notation

$$u_{(1)} = \{u_i^\alpha\}, \quad u_{(2)} = \{u_{ij}^\alpha\}, \ldots$$

with

$$u_i^\alpha = D_i(u^\alpha), \quad u_{ij}^\alpha = D_i(u_j^\alpha) = D_iD_j(u^\alpha), \ldots ,$$

where D_i denotes the total differentiation with respect to x^i :

$$D_i = \frac{\partial}{\partial x^i} + u_i^\alpha \frac{\partial}{\partial u^\alpha} + u_{ij}^\alpha \frac{\partial}{\partial u_j^\alpha} + \cdots \tag{2.1}$$

The variables u^α are also called *differential variables.* A function

$$f(x, u, u_{(1)}, \ldots)$$

of a finite number of variables $x, u, u_{(1)}, u_{(2)}, \ldots$ is called a *differential function* if it is locally analytic, i.e., locally expandable in a Taylor series with respect to all arguments. If the highest order of derivatives appearing in a differential function f is equal to s, i.e.

$$f = f(x, u, u_{(1)}, \ldots, u_{(s)}),$$

we say that f is a differential function of order s and write

$$\mathrm{ord} f = s.$$

The set of all differential functions of all finite orders is denoted by $\mathcal{A}$. This set is a vector space with respect to the usual addition of functions and becomes an associative algebra if multiplication is defined by the usual multiplication of functions. The space $\mathcal{A}$ is closed under the total differentiations: if $f \in \mathcal{A}$ then $D_i(f) \in \mathcal{A}$.

2.2 Basic operators and the fundamental identity

The *Euler-Lagrange operator* (variational derivative) in $\mathcal{A}$ is defined by

$$\frac{\delta}{\delta u^\alpha} = \frac{\partial}{\partial u^\alpha} + \sum_{s=1}^{\infty} (-1)^s D_{i_1} \cdots D_{i_s} \frac{\partial}{\partial u_{i_1 \cdots i_s}^\alpha}, \quad \alpha = 1, \ldots, m, \tag{2.2}$$

where the summation over the repeated indices $i_1 \ldots i_s$ runs from 1 to n.

A first-order linear differential operator

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha} + \zeta_i^\alpha \frac{\partial}{\partial u_i^\alpha} + \zeta_{i_1 i_2}^\alpha \frac{\partial}{\partial u_{i_1 i_2}^\alpha} + \cdots, \tag{2.3}$$

where $\xi^i, \eta^\alpha \in \mathcal{A}$ are arbitrary differential variables, and $\zeta_i^\alpha, \zeta_{i_1 i_2}^\alpha, \ldots$ are given by

$$\zeta_i^\alpha = D_i(W^\alpha) + \xi^j u_{ij}^\alpha, \quad \zeta_{i_1 i_2}^\alpha = D_{i_1} D_{i_2}(W^\alpha) + \xi^j u_{j i_1 i_2}^\alpha, \ldots \tag{2.4}$$

with
$$W^\alpha = \eta^\alpha - \xi^j u_j^\alpha, \quad \alpha = 1, \ldots, m, \tag{2.5}$$
is called a *Lie-Bäcklund operator.* It is often written in the abbreviated form
$$X = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha} + \cdots, \tag{2.6}$$
where the prolongation given by (2.3) - (2.4) is understood. The operator (2.3) is formally an infinite sum, but it truncates when acting on any differential function. Hence, *the action of Lie-Bäcklund operators is well defined on the space* $\mathcal{A}$.

The commutator $[X_1, X_2] = X_1X_2 - X_2X_1$ of any two Lie-Bäcklund operators,
$$X_\nu = \xi_\nu^i \frac{\partial}{\partial x^i} + \eta_\nu^\alpha \frac{\partial}{\partial u^\alpha} + \cdots \quad (\nu = 1, 2),$$
is identical with the Lie-Bäcklund operator given by
$$[X_1, X_2] = \big(X_1(\xi_2^i) - X_2(\xi_1^i)\big)\frac{\partial}{\partial x^i} + \big(X_1(\eta_2^\alpha) - X_2(\eta_1^\alpha)\big)\frac{\partial}{\partial u^\alpha} + \cdots. \tag{2.7}$$

The set of all Lie-Bäcklund operators is an infinite dimensional Lie algebra with respect to the commutator (2.7). It is called the *Lie-Bäcklund algebra* and denoted by $L_\mathcal{B}$. The algebra $L_\mathcal{B}$ is endowed with the following properties.

I. $D_i \in L_\mathcal{B}$. In other words, the total differentiation (2.1) is a Lie-Bäcklund operator. Furthermore,
$$X_* = \xi_*^i D_i \in L_\mathcal{B} \tag{2.8}$$
for any $\xi_*^i \in \mathcal{A}$.

II. Let L_* be the set of all Lie-Bäcklund operators of the form (2.8). Then L_* is an ideal of $L_\mathcal{B}$, i.e., $[X, X_*] \in L_*$ for any $X \in L_\mathcal{B}$. Indeed,
$$[X, X_*] = \big(X(\xi_*^i) - X_*(\xi^i)\big)D_i \in L_*.$$

III. In accordance with property II, two operators $X_1, X_2 \in L_\mathcal{B}$ are said to be *equivalent* (i.e. $X_1 \sim X_2$) if $X_1 - X_2 \in L_*$. In particular, every operator $X \in L_\mathcal{B}$ is equivalent to an operator (2.3) with $\xi^i = 0, i = 1, ..., n$. Namely, $X \sim \widetilde{X}$ where
$$\widetilde{X} = X - \xi^i D_i = (\eta^\alpha - \xi^i u_i^\alpha)\frac{\partial}{\partial u^\alpha} + \cdots. \tag{2.9}$$

The operators of the form

$$X = \eta^\alpha \frac{\partial}{\partial u^\alpha} + \cdots, \quad \eta^\alpha \in \mathcal{A}, \tag{2.10}$$

are called *canonical Lie-Bäcklund operators*. Hence, the property III means that any $X \in L_{\mathcal{B}}$ is equivalent to a canonical Lie-Bäcklund operator.

IV. Generators of Lie point transformation groups are operators (2.6) with the coefficients ξ^i and η^α depending only on x, u :

$$X = \xi^i(x,u)\frac{\partial}{\partial x^i} + \eta^\alpha(x,u)\frac{\partial}{\partial u^\alpha} \cdot \tag{2.11}$$

The Lie-Bäcklund operator (2.3) is equivalent to a generator (2.11) of a point transformation group if and only if its coordinates have the form

$$\xi^i = \xi^i_1(x,u) + \xi^i_*, \quad \eta^\alpha = \eta^\alpha_1(x,u) + \big(\xi^i_2(x,u) + \xi^i_*\big)u^\alpha_i,$$

where $\xi^i_* \in \mathcal{A}$ are any differential functions whereas ξ^i_1, ξ^i_2 and η^α_1 depend only on the variables x, u.

Example 10.1. Let t, x be the independent variables. The generator of the Galilean transformation and its canonical Lie-Bäcklund form (2.9) are written:

$$X = \frac{\partial}{\partial u} - t\frac{\partial}{\partial x} \sim \widetilde{X} = (1 + tu_x)\frac{\partial}{\partial u} + \cdots.$$

Example 10.2. The generator of non-homogeneous dilations (see operator X_2 in Section 4) and its canonical Lie-Bäcklund representation (2.9) are written:

$$X = 2u\frac{\partial}{\partial u} - 3t\frac{\partial}{\partial t} - x\frac{\partial}{\partial x} \sim \widetilde{X} = (2u + 3tu_t + xu_x)\frac{\partial}{\partial u} + \cdots.$$

I associate with Lie-Bäcklund operators $X = \xi^i\frac{\partial}{\partial x^i} + \eta^\alpha\frac{\partial}{\partial u^\alpha} + \cdots$ of the form (2.3) the infinite-order operators $\mathcal{N}^i$ $(i = 1, \ldots, n)$ defined by the following formal sums:

$$\mathcal{N}^i = \xi^i + W^\alpha\,\frac{\delta}{\delta u^\alpha_i} + \sum_{s=1}^{\infty} D_{i_1}\cdots D_{i_s}(W^\alpha)\,\frac{\delta}{\delta u^\alpha_{ii_1\cdots i_s}}, \tag{2.12}$$

where W^α are given by (2.5) and the variational derivatives with respect to variables $u^\alpha_i, \ldots$ are obtained from (2.2) by replacing u^α by the corresponding derivatives $u^\alpha_i, \ldots$, e.g.

$$\frac{\delta}{\delta u^\alpha_i} = \frac{\partial}{\partial u^\alpha_i} + \sum_{s=1}^{\infty}(-1)^s D_{j_1}\cdots D_{j_s}\,\frac{\partial}{\partial u^\alpha_{ij_1\cdots j_s}} \cdot \tag{2.13}$$

In terms of the above operators and constructions, the following statement holds (N.H. Ibragimov, 1979, see [39], Section 8.4.4).

Theorem 10.2. The operators (2.2), (2.3) and (2.12) are connected by the following *fundamental identity:*

$$X + D_i(\xi^i) = W^\alpha \frac{\delta}{\delta u^\alpha} + D_i \mathcal{N}^i\,, \tag{2.14}$$

where W^α are given by (2.5).

2.3 Adjoint operators to linear operators

Recall the classical definition of the adjoint operators to linear operators.

Definition 10.1. Let L be a linear differential operator. A linear operator L^* is called the *adjoint operator* to L if the equation

$$vL[u] - uL^*[v] = D_i(p^i) \tag{2.15}$$

holds with certain functions $p^i \in \mathcal{A}$. The equation $L^*[v] = 0$ is called the adjoint equation to $L[u] = 0$.

Example 10.3. Let L be the linear second-order differential operator:

$$L[u] = a^{ij}(x)D_iD_j(u) + b^i(x)D_i(u) + c(x)u, \quad a^{ji} = a^{ij}. \tag{2.16}$$

Eq. (2.15) yields that the adjoint operator L^* is defined by

$$L^*[v] = D_iD_j(a^{ij}(x)v) - D_i(b^i(x)v) + c(x)v, \tag{2.17}$$

or

$$L^*[v] = a^{ij}v_{ij} + \left[2D_j(a^{ij}) - b^i\right]v_i + \left[c - D_i(b^i) + D_iD_j(a^{ij})\right]v. \tag{2.18}$$

Indeed, we have:

$$\begin{aligned} vLu &= va^{ij}D_iD_j(u) + vb^iD_i(u) + cuv \\ &= D_i[va^{ij}D_j(u)] - D_i(va^{ij})D_j(u) + D_i(vb^iu) - uD_i(vb^i) + ucv. \end{aligned}$$

Furthermore, writing

$$-D_i(va^{ij})D_j(u) = -D_j[uD_i(va^{ij})] + uD_iD_j(a^{ij}v)$$

and taking into account that $a^{ji} = a^{ij}$ and $D_j(u) = u_j$ we obtain:

$$vLu = D_i\{a^{ij}vu_j + b^i uv - uD_j(a^{ij}v)\} + u[D_iD_j(a^{ij}v) - D_i(b^i v) + cv].$$

Substituting the resulting expression for vLu in (2.15) we conclude that the adjoint operator L^* is defined by (2.17) and that Eq. (2.15) is satisfied with

$$p^i = a^{ij}vu_j + b^i uv - uD_j(a^{ij}v).$$

Definition 10.2. The operator L and the equation $L[u] = 0$ are said to be *self-adjoint* if the following equation holds for any function $u(x)$:

$$L^*[u] = L[u]. \tag{2.19}$$

Example 10.4. The expression (2.18) for the adjoint operator shows that the operator (2.16) is self-adjoint if

$$b^i(x) = D_j(a^{ij}), \quad i = 1, \ldots, n. \tag{2.20}$$

The definitions of the adjoint operator and self-adjoint equations for systems of linear differential equations are the same as in the case of scalar equations. For example, in the case of systems of second-order equations the adjoint operator is obtained by assuming that u is an m-dimensional vector-function and that the coefficients $a^{ij}(x), b^i(x)$ and $c(x)$ of the operator (2.16) are $m \times m$-matrices. The following two second-order equations provide an example of a self-adjoint system:

$$x^2u_{xx} + u_{yy} + 2xu_x + w = 0,$$

$$w_{xx} + y^2w_{yy} + 2yw_y + u = 0.$$

2.4 Adjoint equations for arbitrary equations

Linearity of equations is crucial for defining adjoint equations by means of Eq. (2.15). The following definition of an adjoint equation [46] is applicable to any system of linear and non-linear differential equations.

Definition 10.3. Consider a system of partial differential equations

$$F_\alpha\big(x, u, \ldots, u_{(s)}\big) = 0, \quad \alpha = 1, \ldots, m, \tag{2.21}$$

with n independent variables $x = (x^1, \ldots, x^n)$ and m dependent variables $u = (u^1, \ldots, u^m)$. It is assumed that each $F_\alpha(x, u, \ldots, u_{(s)})$ is a differential function such that ord$F_\alpha \leq s$. We introduce the differential functions

$$F^*_\alpha\big(x, u, v, \ldots, u_{(s)}, v_{(s)}\big) = \frac{\delta(v^\beta F_\beta)}{\delta u^\alpha}\,, \quad \alpha = 1, \ldots, m, \tag{2.22}$$

where $v = (v^1, \ldots, v^m)$ are new dependent variables, and define the system of *adjoint equations* to Eqs. (2.21) by

$$F^*_\alpha\big(x, u, v, \ldots, u_{(s)}, v_{(s)}\big) = 0, \quad \alpha = 1, \ldots, m. \tag{2.23}$$

Example 10.5. For the heat equation $u_t - u_{xx} = 0$ Eq. (2.22) yields

$$\begin{aligned} F^* &= \frac{\delta}{\delta u}\big[v\big(v_t - u_{xx}\big)\big] \\ &= \left(-D_t\,\frac{\partial}{\partial u_t} + D_x^2\,\frac{\partial}{\partial u_{xx}}\right)\big[v\big(u_t - u_{xx}\big)\big] = -D_t(v) - D_x^2(v). \end{aligned}$$

Hence, the adjoint equation (2.23) to the heat equation is

$$v_t + v_{xx} = 0.$$

Example 10.6. For the Korteweg-de Vries (KdV) equation

$$u_t = u_{xxx} + uu_x.$$

Eq. (2.22) yields

$$F^*(t, x, u, v, \ldots, u_{(3)}, v_{(3)}) = -(v_t - v_{xxx} - uv_x).$$

Hence, the adjoint equation to the KdV equation is

$$v_t = v_{xxx} + uv_x.$$

In the case of linear equations, Definition 10.3 leads to the same adjoint equation as Definition 10.1. Namely, the following statement holds.

Theorem 10.3. The adjoint operator $L^*[v]$ to a linear operator L defined by Eq. (2.15) is identical with the adjoint operator given by Eq. (2.22), i.e.

$$L^*[v] = \frac{\delta(vL[u])}{\delta u}\,\cdot \tag{2.24}$$

Proof. Let us use for a while the notation

$$\frac{\delta(vL[u])}{\delta u} = \tilde{L}[v].$$

Let $L^*[v]$ be defined by Eq. (2.15). Then, invoking Lemma 10.1, we obtain:

$$\tilde{L}[v] \equiv \frac{\delta(vL[u])}{\delta u} = \frac{\delta}{\delta u}\Big(uL^*[v] + D_i(p^i)\Big)$$
$$= \frac{\delta}{\delta u}\Big(uL^*[v]\Big) = \frac{\partial}{\partial u}\Big(uL^*[v]\Big) = L^*[v].$$

Conversely, let us verify that $\tilde{L}[v]$ satisfies Eq. (2.15), i.e.

$$vL[u] - u\tilde{L}[v] = D_i(\tilde{p}^{\,i})$$

with certain $\tilde{p}^{\,i} \in \mathcal{A}$. According to Lemma 10.1, we have to show that

$$\frac{\delta}{\delta u}\Big(vL[u] - u\tilde{L}[v]\Big) = 0. \tag{2.25}$$

Invoking the definition of $\tilde{L}[v]$ we have:

$$\frac{\delta}{\delta u}\Big(vL[u] - u\tilde{L}[v]\Big) = \frac{\delta(vL[u])}{\delta u} - \frac{\delta\big(u\tilde{L}[v]\big)}{\delta u} = \tilde{L}[v] - \frac{\delta\big(u\tilde{L}[v]\big)}{\delta u}\,\cdot \tag{2.26}$$

Since $\tilde{L}[v]$ does not depend on the variables $u, u_i, \ldots$, we obtain:

$$\frac{\delta\big(u\tilde{L}[v]\big)}{\delta u} = \Big(\frac{\partial}{\partial u} - D_i\frac{\partial}{\partial u_i} + \cdots\Big)\big(u\tilde{L}[v]\big) = \tilde{L}[v].$$

Substituting this in Eq. (2.26) we obtain Eq. (2.25). Theorem is proved.

Remark 10.2. In practice, the calculation of the adjoint operator by (2.22) is simpler than by (2.15).

Example 10.7. Let us illustrate Theorem 10.3 for the second-order linear operator (2.16). In other words, let us show that the expression for $L^*[v]$ defined by Eq. (2.24) coincides with $L^*[v]$ defined by Eq. (2.17). Thus, we consider the second-order linear partial differential equation

$$F\big(x, u, u_{(1)}, u_{(2)}\big) \equiv a^{ij}(x)u_{ij} + b^i(x)u_i + cu = 0.$$

Eq. (2.22) is written:

$$F^* = \left(\frac{\partial}{\partial u} - D_i \frac{\partial}{\partial u_i} + D_i D_j \frac{\partial}{\partial u_{ij}}\right)\Big(v[a^{ij}(x)u_{ij} + b^i(x)u_i + cu]\Big)$$

and yields the adjoint operator (2.17):

$$L^*[v] \equiv F^* = D_i D_j(a^{ij}v) - D_i(b^i v) + cv.$$

Remark 10.3. The adjoint equations (2.23) to linear equations

$$F(x, u, \ldots, u_{(s)}) = 0$$

for $u(x)$ provide linear equations

$$F^*(x, v, \ldots, v_{(s)}) = 0$$

for $v(x)$. If Eqs. (2.21) are nonlinear, the adjoint equations are linear with respect to $v(x)$, but nonlinear in the coupled variables u and v.

Definition 10.2 of self-adjointness of linear operators can be extended to nonlinear equations as follows [46].

Definition 10.4. A system of equations (2.21) is said to be *self-adjoint* if the system obtained from the adjoint equations (2.23) by setting $v = u$,

$$F^*_\alpha\big(x, u, u, \ldots, u_{(s)}, u_{(s)}\big) = 0, \quad \alpha = 1, \ldots, m, \tag{2.27}$$

is identical with the original system (2.21). In other words, the *self-adjoint* equations obey the condition (cf. Eq. (2.19))

$$F^*_\alpha(x, u, u, \ldots, u_{(s)}, u_{(s)}) = \phi_\alpha^\beta\, F_\alpha(x, u, \ldots, u_{(s)}), \quad \alpha = 1, \ldots, m, \tag{2.28}$$

with regular coefficients $\phi_\alpha^\beta \in \mathcal{A}$.

Example 10.8. By Example 10.5, the heat equation is not self-adjoint.

Example 10.9. Example 10.6 shows that the KdV equation is self-adjoint. Indeed, it is manifest that the adjoint equation $v_t = v_{xxx} + uv_x$ coincides with the KdV equation upon the substitution $v = u$. We see from calculations in Example 10.6 that

$$F^*(t, x, u, u, \ldots, u_{(3)}, u_{(3)}) = -F(t, x, u, \ldots, u_{(3)}),$$

and hence the condition (2.28) is satisfied with $\phi = -1$.

2.5 Formal Lagrangians

Consider the extension of the variational derivatives (2.2) to differential functions with $2m$ differential variables $(u, v) = (u^1, \ldots, u^m; v^1, \ldots, v^m)$ defined by the coupled formal sums:

$$\frac{\delta}{\delta u^\alpha} = \frac{\partial}{\partial u^\alpha} + \sum_{s=1}^{\infty}(-1)^s D_{i_1} \cdots D_{i_s} \frac{\partial}{\partial u^\alpha_{i_1 \cdots i_s}}\,,$$

$$\frac{\delta}{\delta v^\alpha} = \frac{\partial}{\partial v^\alpha} + \sum_{s=1}^{\infty}(-1)^s D_{i_1} \cdots D_{i_s} \frac{\partial}{\partial v^\alpha_{i_1 \cdots i_s}}\,, \tag{2.29}$$

where $\alpha = 1, \ldots, m$ and the summation indices $i_1 \ldots i_s$ run from 1 to n.

Consider the differential function

$$\mathcal{L} = v^\beta F_\beta\big(x, u, \ldots, u_{(s)}\big) \tag{2.30}$$

with $2m$ differential variables (u, v), where

$$u = (u^1, \ldots, u^m), \quad v = (v^1, \ldots, v^m).$$

It is manifest from Eqs. (2.22) that the variational derivatives (2.29) of the function (2.30) provide the differential equations (2.21) and their adjoint equations (2.23), namely:

$$\frac{\delta \mathcal{L}}{\delta v^\alpha} = F_\alpha\big(x, u, \ldots, u_{(s)}\big), \tag{2.31}$$

$$\frac{\delta \mathcal{L}}{\delta u^\alpha} = F^*_\alpha\big(x, u, v, \ldots, u_{(s)}, v_{(s)}\big). \tag{2.32}$$

This circumstance justifies the following definition.

Definition 10.5. The differential function (2.30) is called a *formal Lagrangian* for the differential equations (2.21).

Example 10.10. The heat equation $u_t - u_{xx} = 0$ has the *second-order* formal Lagrangian

$$\mathcal{L} = v\big(u_t - u_{xx}\big).$$

Using Lemma 10.1 and the identity $-vu_{xx} = (-vu_x)_x + u_x v_x$, one can replace it by the equivalent *first-order* formal Lagrangian

$$\mathcal{L} = vu_t + u_x v_x.$$

The variational derivatives (2.29) of both $\mathcal{L}$ provide the heat equation together with its adjoint equation $v_t + v_{xx} = 0$:

$$\frac{\delta \mathcal{L}}{\delta v} = u_t - u_{xx}, \quad \frac{\delta \mathcal{L}}{\delta u} = -(v_t + v_{xx}).$$

Let us extend Example 10.10 to the general linear second-order equation

$$L[u] \equiv a^{ij}(x)u_{ij} + b^i(x)u_i + c(x)u = 0. \tag{2.33}$$

The formal Lagrangian (2.30) is written

$$\mathcal{L} = \big(a^{ij}(x)u_{ij} + b^i(x)u_i + c(x)u\big)v.$$

We rewrite it in the form

$$\mathcal{L} = D_j\big(va^{ij}u_i\big) - vu_iD_j\big(a^{ij}\big) - a^{ij}u_iv_j + vb^iu_i + cuv.$$

We drop the first term at the right-hand side by Lemma 10.1 and obtain

$$\mathcal{L} = cuv + vb^i(x)u_i - vu_iD_j\big(a^{ij}\big) - a^{ij}u_iv_j. \tag{2.34}$$

The variational differentiation of the function (2.34) leads to Eq. (2.33) and its adjoint equation $D_iD_j\big(a^{ij}v\big) - D_i\big(b^iv\big) + cv = 0$:

$$\frac{\delta \mathcal{L}}{\delta v} = cu + b^i(x)u_i - u_iD_j\big(a^{ij}\big) + D_j\big(a^{ij}u_i\big) = a^{ij}u_{ij} + b^iu_i + cu,$$

$$\frac{\delta \mathcal{L}}{\delta u} = cv - D_i(b^iv) + D_i\big(vD_j(a^{ij}v)\big) + D_i(a^{ij}v_j) = D_iD_j(a^{ij}v) - D_i(b^iv) + cv.$$

Example 10.11. If the operator $L[u]$ is self-adjoint (see the condition (2.20)), then (2.34) yields to the following well known Lagrangian for Eq. (2.33):

$$\mathcal{L} = \frac{1}{2}\big[c(x)u^2 - a^{ij}(x)u_iu_j\big] \tag{2.35}$$

Indeed, the second and the third terms in the right-hand side of Eq. (2.34) annihilate each other by the condition (2.20). Setting $v = u$ and dividing the resulting expression by two we obtain the Lagrangian (2.35). The Lagrangian for the the non-homogeneous equation $L[u] = f(x)$ with self-adjoint operator L is obtained from (2.35) by adding the term $-f(x)u$.

Example 10.12. According to Example 10.6, the KdV equation

$$u_t = u_{xxx} + uu_x \tag{2.36}$$

has the following adjoint equation:

$$v_t = v_{xxx} + uv_x. \tag{2.37}$$

Eq. (2.30) yields the third-order formal Lagrangian for the KdV equation:

$$\mathcal{L} = v[u_t - uu_x - u_{xxx}]. \tag{2.38}$$

Using the equation $-vu_{xxx} = (-vu_{xx})_x + v_x u_{xx}$ and Lemma 10.1, one can replace (2.38) by the second-order formal Lagrangian

$$\mathcal{L} = vu_t - vuu_x + v_x u_{xx}. \tag{2.39}$$

Furthermore, one can easily replace (2.39) by

$$\mathcal{L} = v_x u_{xx} - uv_t + \frac{1}{2}u^2 v_x. \tag{2.40}$$

Each of the functions (2.38), (2.39) and (2.40) yield the KdV equation and its adjoint:

$$\frac{\delta \mathcal{L}}{\delta v} = u_t - uu_x - u_{xxx}, \quad \frac{\delta \mathcal{L}}{\delta u} = -v_t + v_{xxx} + uv_x.$$

3 Main theorems

3.1 Symmetry of adjoint equations

Let us show that the adjoint equations (2.23) inherit all Lie and Lie-Bäcklund symmetries of Eqs. (2.21). We will begin with scalar equations.

Theorem 10.4. Consider an equation

$$F\big(x, u, u_{(1)}, \ldots, u_{(s)}\big) = 0 \tag{3.1}$$

with n independent variables $x = (x^1, \ldots, x^n)$ and one dependent variable u. The adjoint equation

$$F^*\big(x, u, v, \ldots, u_{(s)}, v_{(s)}\big) \equiv \frac{\delta(vF)}{\delta u} = 0 \tag{3.2}$$

to Eq. (3.1) inherits the symmetries of Eq. (3.1). Namely, if Eq. (3.1) admits an operator

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta \frac{\partial}{\partial u}\,, \tag{3.3}$$

where X is either a generator of a point transformation group, i.e.

$$\xi^i = \xi^i(x, u), \quad \eta = \eta(x, u),$$

or a Lie-Bäcklund operator, i.e.

$$\xi^i = \xi^i(x, u, u_{(1)}, \ldots, u_{(p)}), \quad \eta = \eta(x, u, u_{(1)}, \ldots, u_{(q)}),$$

then Eq. (3.2) admits the operator (3.3) extended to the variable v by the formula

$$Y = \xi^i \frac{\partial}{\partial x^i} + \eta \frac{\partial}{\partial u} + \eta_* \frac{\partial}{\partial v} \tag{3.4}$$

with a certain function $\eta_* = \eta_*(x, u, v, u_{(1)}, \ldots) \in \mathcal{A}$.

Proof. Let the operator (3.3) be a Lie point symmetry of Eq. (3.1). Then

$$X(F) = \lambda\, F \tag{3.5}$$

where $\lambda = \lambda(x, u, \ldots)$. In Eq. (3.5), the prolongation of X to all derivatives involved in Eq. (3.1) is understood. Furthermore, the simultaneous system (3.1), (3.2) can be obtained as the variational derivatives of the formal Lagrangian (2.30)

$$\mathcal{L} = vF \tag{3.6}$$

of Eq. (3.1). We take an extension of the operator (3.3) in the form (3.4) with an unknown coefficient η_* and require that the invariance condition (1.10) be satisfied:

$$Y(\mathcal{L}) + \mathcal{L} D_i(\xi^i) = 0. \tag{3.7}$$

We have:

$$Y(\mathcal{L}) + \mathcal{L} D_i(\xi^i) = Y(v)F + vX(F) + vFD_i(\xi^i)$$

$$= \eta_* F + v\lambda F + vFD_i(\xi^i) = [\eta_* + v\lambda + vD_i(\xi^i)]F.$$

Hence, the requirement (3.7) leads to the equation

$$\eta_* = -\big[\lambda + D_i(\xi^i)\big]v. \tag{3.8}$$

with λ defined by Eq. (3.5). Since Eq. (3.7) guarantees the invariance of the system (3.1), (3.2) we conclude that the adjoint equation (3.2) admits the operator

$$Y = \xi^i \frac{\partial}{\partial x^i} + \eta \frac{\partial}{\partial u} - \big[\lambda + D_i(\xi^i)\big] v \, \frac{\partial}{\partial v} \tag{3.9}$$

thus proving the theorem for Lie point symmetries.

Let us assume now that the symmetry (3.3) is a Lie-Bäcklund operator. Then Eq. (3.5) is replaced by (see [34])

$$X(F) = \lambda_0 \, F + \lambda_1^i \, D_i(F) + \lambda_2^{ij} \, D_i D_j(F) + \lambda_3^{ij} \, D_i D_j D_k(F) + \cdots \, , \tag{3.10}$$

where $\lambda_2^{ij} = \lambda_2^{ji}, \ldots$. Therefore, using the operator (3.4), we have:

$$Y(\mathcal{L}) + \mathcal{L} D_i(\xi^i) = Y(v) F + v X(F) + v F D_i(\xi^i)$$

$$= [\eta_* + v\lambda_0 + v D_i(\xi^i)] F + v\lambda_1^i \, D_i(F) + v\lambda_2^{ij} \, D_i D_j(F) + v\lambda_3^{ijk} \, D_i D_j D_k(F) + \cdots \, .$$

Now we use the identities

$$v\lambda_1^i \, D_i(F) = D_i\big(v\lambda_1^i \, F\big) - F D_i\big(v\lambda_1^i\big),$$

$$v\lambda_2^{ij} \, D_i D_j(F) = D_i\big[v\lambda_2^{ij} D_j(F) - F D_j(v\lambda_2^{ij})\big] + F D_i D_j\big(v\lambda_2^{ij}\big),$$

$$v\lambda_3^{ijk} \, D_i D_j D_k(F) = D_i\big[\cdots\big] - F D_i D_j D_k\big(v\lambda_3^{ijk}\big),$$

etc., and obtain:

$$Y(\mathcal{L}) + \mathcal{L} D_i(\xi^i) = D_i\big[v\lambda_1^i \, F + v\lambda_2^{ij} D_j(F) - F D_j(v\lambda_2^{ij}) + \cdots\big]$$

$$+ \,[\eta_* + v\lambda_0 + v D_i(\xi^i) - D_i\big(v\lambda_1^i\big) + D_i D_j\big(v\lambda_2^{ij}\big) - D_i D_j D_k\big(v\lambda_3^{ijk}\big) + \cdots]\, F.$$

Finally, we complete the proof of the theorem by setting

$$\eta_* = -\big[\lambda_0 + D_i(\xi^i)\big] v + D_i\big(v\lambda_1^i\big) - D_i D_j\big(v\lambda_2^{ij}\big) + D_i D_j D_k\big(v\lambda_3^{ijk}\big) - \cdots \tag{3.11}$$

and arriving at Eq. (1.14),

$$X(\mathcal{L}) + \mathcal{L} D_i(\xi^i) = D_i(B^i), \tag{1.14}$$

with

$$B^i = -v\lambda_1^i \, F - v\lambda_2^{ij} D_j(F) + F D_j(v\lambda_2^{ij}) - \cdots \tag{3.12}$$

Let us prove a similar statement on symmetries of adjoint equations for systems of m equations with m dependent variables. For the sake of simplicity we will prove the theorem only for Lie point symmetries. The proof can be extended to Lie-Bäcklund symmetries as it has been done in Theorem 10.4.

Theorem 10.5. Consider a system of m equations

$$F_\alpha\big(x, u, u_{(1)}, \ldots, u_{(s)}\big) = 0, \quad \alpha = 1, \ldots, m, \tag{3.13}$$

with n independent variables $x = (x^1, \ldots, x^n)$ and m dependent variables $u = (u^1, \ldots, u^m)$. The adjoint system

$$F_\alpha^*\big(x, u, v, \ldots, u_{(s)}, v_{(s)}\big) \equiv \frac{\delta(v^\beta F_\beta)}{\delta u^\alpha} = 0, \quad \alpha = 1, \ldots, m, \tag{3.14}$$

inherits the symmetries of the system (3.13). Namely, if the system (3.13) admits a point transformation group with a generator

$$X = \xi^i(x, u)\frac{\partial}{\partial x^i} + \eta^\alpha(x, u)\frac{\partial}{\partial u^\alpha}\,, \tag{3.15}$$

then the adjoint system (3.14) admits the operator (3.15) extended to the variables v^α by the formula

$$Y = \xi^i\frac{\partial}{\partial x^i} + \eta^\alpha\frac{\partial}{\partial u^\alpha} + \eta_*^\alpha\frac{\partial}{\partial v^\alpha} \tag{3.16}$$

with appropriately chosen coefficients $\eta_*^\alpha = \eta_*^\alpha(x, u, v, \ldots)$.

Proof. Now the invariance condition (3.5) is replaced by

$$X(F_\alpha) = \lambda_\alpha^\beta F_\beta, \quad \alpha = 1, \ldots, m, \tag{3.17}$$

where the prolongation of X to all derivatives involved in Eqs. (3.13) is understood. We know that the simultaneous system (3.13), (3.14) can be obtained as the variational derivatives of the formal Lagrangian

$$\mathcal{L} = v^\alpha F_\alpha \tag{3.18}$$

of Eqs. (3.13). We take an extension of the operator (3.15) in the form (3.16) with undetermined coefficients η_*^α and require that the invariance condition (1.10) be satisfied:

$$Y(\mathcal{L}) + \mathcal{L}D_i(\xi^i) = 0. \tag{3.19}$$

We have:

$$Y(\mathcal{L}) + \mathcal{L}D_i(\xi^i) = Y(v^\alpha)F_\alpha + v^\alpha X(F_\alpha) + v^\alpha F_\alpha D_i(\xi^i)$$

$$= \eta_*^\alpha F_\alpha + \lambda_\alpha^\beta v^\alpha F_\beta + v^\alpha F_\alpha D_i(\xi^i) = [\eta_*^\alpha + \lambda_\beta^\alpha v^\beta + v^\alpha D_i(\xi^i)]F_\alpha.$$

Therefore, the requirement (3.19) leads to the equations

$$\eta_*^\alpha = -\big[\lambda_\beta^\alpha v^\beta + v^\alpha D_i(\xi^i)\big], \quad \alpha = 1, \ldots, m, \tag{3.20}$$

with λ_β^α defined by Eqs. (3.17). Since Eqs. (3.19) guarantee the invariance of the system (3.13), (3.14), the adjoint system (3.14) admits the operator

$$Y = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha} - \big[\lambda_\beta^\alpha v^\beta + v^\alpha D_i(\xi^i)\big] \frac{\partial}{\partial v^\alpha} \cdot \tag{3.21}$$

This proves the theorem.

Theorem 10.6. Theorems 10.4 and 10.5 are valid for nonlocal symmetries defined in [1].

Proof. The statement is proved by adding to Eqs. (2.21), (2.23) the differential equations defining nonlocal variables involved in nonlocal symmetries and repeating the proofs of Theorems 10.4 and 10.5.

3.2 Theorem on nonlocal conservation laws

Theorem 10.7. Every Lie point and Lie-Bäcklund symmetry

$$X = \xi^i(x, u, u_{(1)}, \ldots) \frac{\partial}{\partial x^i} + \eta^\alpha(x, u, u_{(1)}, \ldots) \frac{\partial}{\partial u^\alpha}\,, \tag{3.22}$$

as well as nonlocal symmetry, of differential equations

$$F_\alpha\big(x, u, \ldots, u_{(s)}\big) = 0, \quad \alpha = 1, \ldots, m, \tag{3.23}$$

provides a *nonlocal conservation law* for Eqs. (3.23). The corresponding conserved quantity involves the adjoint (i.e. nonlocal) variables v given by the adjoint equations (3.14), and hence the resulting conservation laws is, in general, *nonlocal.*

Proof. We take the extended action (3.16) of the operator (3.22),

$$Y = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha} + \eta_*^\alpha \frac{\partial}{\partial v^\alpha} \cdot \tag{3.24}$$

In the case of one variable u, the coefficient η_* in (3.24) is given by (3.8) if (3.3) is a Lie point symmetry and by (3.11) if (3.3) is Lie-Bäcklund symmetry. In the case of several variables u^α and Lie point symmetries (3.15), the

η_*^α are given by (3.20). Now we extend the action of the operators (2.12) to differential functions of $2m$ variables u^α, v^α and obtain the following prolongation of the operators $\mathcal{N}^i$:

$$\widetilde{\boldsymbol{\mathcal{N}}}^i = \xi^i + W^\alpha \frac{\delta}{\delta u_i^\alpha} + W_*^\alpha \frac{\delta}{\delta v_i^\alpha}$$

$$+ \sum_{s=1}^{\infty} \Big[D_{i_1} \cdots D_{i_s}(W^\alpha) \frac{\delta}{\delta u_{ii_1\cdots i_s}^\alpha} + D_{i_1} \cdots D_{i_s}(W_*^\alpha) \frac{\delta}{\delta v_{ii_1\cdots i_s}^\alpha}\Big], \qquad (3.25)$$

where (see (2.5))

$$W^\alpha = \eta^\alpha - \xi^j u_j^\alpha, \quad W_*^\alpha = \eta_*^\alpha - \xi^j v_j^\alpha, \qquad (3.26)$$

and (see (2.13))

$$\frac{\delta}{\delta u_i^\alpha} = \frac{\partial}{\partial u_i^\alpha} - D_{j_1} \frac{\partial}{\partial u_{ij_1}^\alpha} + D_{j_1} D_{j_2} \frac{\partial}{\partial u_{ij_1j_2}^\alpha} - \cdots ,$$

$$\frac{\delta}{\delta v_i^\alpha} = \frac{\partial}{\partial v_i^\alpha} - D_{j_1} \frac{\partial}{\partial v_{ij}^\alpha} + D_{j_1} D_{j_2} \frac{\partial}{\partial v_{ij_1j_2}^\alpha} - \cdots ,$$

$$\frac{\delta}{\delta u_{ii_1}^\alpha} = \frac{\partial}{\partial u_{ii_1}^\alpha} - D_{j_1} \frac{\partial}{\partial u_{ii_1j_1}^\alpha} + D_{j_1} D_{j_2} \frac{\partial}{\partial u_{ii_1j_1j_2}^\alpha} - \cdots , \qquad (3.27)$$

$$\frac{\delta}{\delta v_{ii_1}^\alpha} = \frac{\partial}{\partial v_{ii_1}^\alpha} - D_{j_1} \frac{\partial}{\partial v_{ii_1j_1}^\alpha} + D_{j_1} D_{j_2} \frac{\partial}{\partial v_{ii_1j_1j_2}^\alpha} - \cdots ,$$

. . . .

The fundamental identity (2.14) is extended likewise and has the form:

$$Y + D_i(\xi^i) = W^\alpha \frac{\delta}{\delta u^\alpha} + W_*^\alpha \frac{\delta}{\delta v^\alpha} + D_i \widetilde{\boldsymbol{\mathcal{N}}}^i . \qquad (3.28)$$

Now we act by the both sides of the operator identity (3.28) on the formal Lagrangian (3.18), invoke the equations (3.19) and (2.31)-(2.32) obtain the conservation law

$$D_i(C^i)\Big|_{(3.23),(3.14)} = 0, \qquad (3.29)$$

where

$$C^i = \widetilde{\boldsymbol{\mathcal{N}}}^i(\mathcal{L}). \qquad (3.30)$$

Since the differential function

$$\mathcal{L} = v^\beta F_\beta$$

given by (3.18) does not contain derivatives of the variables v^α, it follows from Eqs. (3.27) that (3.30) reduces to the form

$$C^i = \mathcal{N}^i(\mathcal{L}). \tag{3.31}$$

Substituting in (3.31) the expansion (2.12) of $\mathcal{N}^i$ up to $s = 2$, we obtain the following coordinates of the nonlocal conserved vector:

$$C^i = \xi^i \mathcal{L} + W^\alpha \Big[\frac{\partial \mathcal{L}}{\partial u_i^\alpha} - D_j\Big(\frac{\partial \mathcal{L}}{\partial u_{ij}^\alpha}\Big) + D_j D_k\Big(\frac{\partial \mathcal{L}}{\partial u_{ijk}^\alpha}\Big) - \ldots\Big] \tag{3.32}$$

$$+ D_j\big(W^\alpha\big) \Big[\frac{\partial \mathcal{L}}{\partial u_{ij}^\alpha} - D_k\Big(\frac{\partial \mathcal{L}}{\partial u_{ijk}^\alpha}\Big) + \ldots\Big] + D_j D_k\big(W^\alpha\big)\Big[\frac{\partial \mathcal{L}}{\partial u_{ijk}^\alpha} - \ldots\Big].$$

Note that if (3.3) is a Lie-Bäcklund symmetry, we have Eq. (1.14) instead of Eq. (3.19), and therefore we should add to (3.31) the term $-B^i$ (cf. (1.15)). However, one can ignore this term since the vector B^i defined by (3.12) vanishes on the solutions of Eq. (3.23) (see Remark 10.1). Finally, we complete the proof by invoking Theorem 10.6.

Remark 10.4. Eq. (3.32) shows that for computing nonlocal conserved vectors by using formal Lagrangians in the form (2.12),

$$\mathcal{L} = v^\beta F_\beta,$$

we do not need the expressions W_*^α, and hence the coefficient η_* of the extended operator (3.24).

Remark 10.5. If one changes a formal Lagrangian

$$\mathcal{L} = v^\beta F_\beta$$

to an equivalent form, e.g. as in Examples 10.10 and 10.12, one arrives at a formal Lagrangian containing derivatives of v^α. Then one should use Eq. (3.30) instead of (3.31), and hence one should calculate the coefficient η_* of the operator (3.24).

3.3 An example on Theorems 10.4 and 10.7

Let us consider the heat equation

$$u_t - u_{xx} = 0$$

together with its formal Lagrangian

$$\mathcal{L} = v(u_t - u_{xx})$$

(see Example 10.10) and apply Theorems 10.4 and 10.7 to a Lie point symmetry and a Lie-Bäcklund symmetry. Since we have one dependent variable ($m = 1$) and deal with a second-order formal Lagrangian, Eqs. (3.32) for computing the conserved vectors are written:

$$C^i = \xi^i \mathcal{L} + W\Big[\frac{\partial \mathcal{L}}{\partial u_i} - D_j\Big(\frac{\partial \mathcal{L}}{\partial u_{ij}}\Big)\Big] + D_j(W)\,\frac{\partial \mathcal{L}}{\partial u_{ij}}. \tag{3.33}$$

As an example of a Lie point symmetry I will take the generator

$$X = 2t\frac{\partial}{\partial x} - xu\frac{\partial}{\partial u} \tag{3.34}$$

of the Galilean transformation admitted by the heat equation. Let us extend the operator (3.34) to the variable v by means of Theorem 10.4so that the extended generator will be admitted by the adjoint equation

$$v_t + v_{xx}$$

to the heat equation. The prolongation of X to the derivatives involved in the heat equation has the form

$$X = 2t\frac{\partial}{\partial x} - xu\frac{\partial}{\partial u} - \big(xu_t + 2u_x\big)\frac{\partial}{\partial u_t} - \big(2u_x + xu_{xx}\big)\frac{\partial}{\partial u_{xx}}\cdot$$

The reckoning shows that Eq. (3.5) is written

$$X(u_t - u_{xx}) = -x(u_t - u_{xx}),$$

hence $\lambda = -x$. Noting that in our case $D_i(\xi^i) = 0$ and using (3.8) we obtain $\eta_* = xv$. Hence, the extension (3.9) of the operator (3.34) to v has the form

$$Y = 2t\frac{\partial}{\partial x} - xu\frac{\partial}{\partial u} + xv\frac{\partial}{\partial v}\cdot \tag{3.35}$$

One can readily verify that it is admitted by the system

$$u_t - u_{xx} = 0, \quad v_t + v_{xx} = 0.$$

Let us find the conservation law provided by the symmetry (3.34). Denoting $t = x^1, x = x^2$, we have for the extended operator (3.35):

$$\xi^1 = 0, \quad \xi^2 = 2t, \quad \eta = -xu, \quad \eta_* = xv, \quad W = -(xu + 2tu_x). \tag{3.36}$$

Substituting in (3.33)

$$\mathcal{L} = v(u_t - u_{xx})$$

and invoking Eqs. (3.36) we obtain the conservation equation

$$D_t(C^1) + D_x(C^2) = 0$$

for the vector $C = (C^1, C^2)$ with

$$C^1 = W\frac{\partial \mathcal{L}}{\partial u_t} = vW, \quad C^2 = 2t\mathcal{L} + WD_x(v) - vD_x(W).$$

Substituting here the expressions for $\mathcal{L}$ and W, we get:

$$\begin{aligned} C^1 &= -v(xu + 2tu_x),\\ C^2 &= v(2tu_t + u + xu_x) - (xu + 2tu_x)v_x. \end{aligned} \tag{3.37}$$

This vector involves an arbitrary solution v of the adjoint equation

$$v_t + v_{xx} = 0.$$

Since the adjoint equation does not involve u, we can substitute in (3.37) any solution of the adjoint equation and obtain an infinite number of conservation laws for the heat equation. Let us take, e.g. the solutions $v = -1,\ v = -x$ and $v = -\mathrm{e}^t \sin x$. In the first case, we have:

$$C^1 = xu + 2tu_x, \quad C^2 = -(2tu_t + u + xu_x).$$

Noting that

$$D_t(2tu_x) = D_tD_x(2tu) = D_xD_t(2tu) = D_x(2u + 2tu_t)$$

we can transfer the term $2tu_x$ from C^1 to C^2 in the form $2u + 2tu_t$. Then the components of the conserved vector are written simply

$$C^1 = xu,\ C^2 = u - xu_x.$$

In the second case, $v = -x$, we obtain the vector

$$C^1 = x^2 u + 2txu_x, \quad C^2 = (2t - x^2)u_x - 2txu_t,$$

and simplifying it as before arrive at

$$C^1 = (x^2 - 2t)u, \; C^2 = (2t - x^2)u_x + 2xu.$$

In the case

$$v = -\mathrm{e}^t \sin x,$$

we note that

$$2tu_x \mathrm{e}^t \sin x = D_x(2tu\mathrm{e}^t \sin x) - 2tu\mathrm{e}^t \cos x$$

and simplifying as before obtain:

$$C^1 = \mathrm{e}^t(x \sin x - 2t \cos x)u,$$
$$C^2 = (u + 2tu - xu_x)\mathrm{e}^t \sin x + (xu + 2tu_x)\mathrm{e}^t \cos x.$$

The heat equation has also Lie-Bäcklund symmetries. One of them is

$$X = \big(xu_{xx} + 2tu_{xxx}\big)\frac{\partial}{\partial u} \tag{3.38}$$

We prolong (3.38) to u_t and u_{xx}, denote the prolonged operator again by X and obtain

$$X(u_t - u_{xx}) = xD_x^2(u_t - u_{xx}) + 2tD_x^3(u_t - u_{xx}).$$

It follows that Eq. (3.10) is satisfied and that the only non-vanishing coefficients in (3.10) are

$$\lambda_2^{22} = x, \quad \lambda_3^{222} = 2t.$$

Accordingly, Eq. (3.11) yields:

$$\eta_* = -D_x^2(xv) + D_x^3(2tv) = -2v_x - xv_{xx} + 2tv_{xxx},$$

and hence the extension (3.4) of the operator (3.38) to the variable v is

$$Y = \eta\frac{\partial}{\partial u} + \eta_*\frac{\partial}{\partial v} \equiv \big(xu_{xx} + 2tu_{xxx}\big)\frac{\partial}{\partial u} + \big(2tv_{xxx} - 2v_x - xv_{xx}\big)\frac{\partial}{\partial v}\cdot \tag{3.39}$$

For the operator (3.39), we have

$$\xi^1 = \xi^2 = 0, \quad W = \eta = xu_{xx} + 2tu_{xxx}.$$

Therefore, (3.33) provides the conserved vector with the following components:

$$C^1 = W\frac{\partial \mathcal{L}}{\partial u_t} = \eta v = [xu_{xx} + 2tu_{xxx}]v,$$

$$C^2 = 2t\mathcal{L} - WD_x\Big(\frac{\partial \mathcal{L}}{\partial u_{xx}}\Big) + D_x(W)\frac{\partial \mathcal{L}}{\partial u_{xx}} = [2t(u_t - u_{xx}) - D_x(\eta)]v + \eta D_x(v)$$
$$= [2t(u_t - u_{xx} - u_{xxxx}) - u_{xx} - xu_{xxx}]v + [u_{xx} + 2tu_{xxx}]v_x.$$

4 Application to the KdV equation

4.1 Generalities

The KdV equation (2.36),

$$u_t = u_{xxx} + uu_x, \tag{2.36}$$

has the formal Lagrangian (2.38),

$$\mathcal{L} = v[u_t - uu_x - u_{xxx}]. \tag{2.38}$$

Since we have one dependent variable ($m = 1$) and deal with a third-order formal Lagrangian, Eqs. (3.32) for computing the conserved vectors are written:

$$C^i = \xi^i\mathcal{L} + W\Big[\frac{\partial \mathcal{L}}{\partial u_i} - D_j\Big(\frac{\partial \mathcal{L}}{\partial u_{ij}}\Big) + D_jD_k\Big(\frac{\partial \mathcal{L}}{\partial u_{ijk}}\Big)\Big]$$
$$+ D_j(W)\,\Big[\frac{\partial \mathcal{L}}{\partial u_{ij}} - D_k\Big(\frac{\partial \mathcal{L}}{\partial u_{ijk}}\Big)\Big] + D_jD_k(W)\,\frac{\partial \mathcal{L}}{\partial u_{ijk}}\,\cdot \tag{4.1}$$

As before, I will set $t = x^1, x = x^2$ and write the conservation equation in the form

$$D_t(C^1) + D_x(C^2) = 0.$$

Let us begin by applying Theorem 10.7 to the generators X_1 and X_2 of the Galilean and scaling transformations, respectively:

$$X_1 = \frac{\partial}{\partial u} - t\frac{\partial}{\partial x}\,, \quad X_2 = 2u\frac{\partial}{\partial u} - 3t\frac{\partial}{\partial t} - x\frac{\partial}{\partial x}\,,$$

which are obviously admitted by the KdV equation.

The operator X_1 yields the conservation law $D_t(C^1)+D_x(C^2)=0$, where the conserved vector $C=(C^1,C^2)$ is given by (4.1) and has the components

$$C^1 = (1+tu_x)v,$$
$$C^2 = t(v_x u_{xx} - u_x v_{xx} - vu_t) - uv - v_{xx}.$$

Since the KdV equation is self-adjoint (see Example 10.12), we let $v=u$, transfer the term

$$tuu_x = D_x(\frac{1}{2}tu^2)$$

from C^1 to C^2 in the form

$$tuu_t + \frac{1}{2}u^2$$

and obtain

$$C^1 = u, \quad C^2 = -\frac{1}{2}u^2 - u_{xx}. \tag{4.2}$$

Let us make more detailed calculations for the operator X_2. For this operator, we have

$$W = (2u + 3tu_t + xu_x)$$

and the vector (4.1) is written:

$$C^1 = -3t\mathcal{L} + Wv = (3tu_{xxx} + 3tuu_x + xu_x + 2u)v,$$
$$C^2 = -x\mathcal{L} - (uv + v_{xx})W + v_x D_x(W) - vD_x^2(W) = -(2u^2 + xu_t + 3tuu_t$$
$$+ 4u_{xx} + 3tu_{txx})v + (3u_x + 3tu_{tx} + xu_{xx})v_x - (2u + 3tu_t + xu_x)v_{xx}.$$

As before, we let $v=u$, simplify the conserved vector by transferring the terms of the form $D_x(\ldots)$ from C^1 to C^2 and obtain

$$C^1 = u^2, \quad C^2 = u_x^2 - 2uu_{xx} - \frac{2}{3}\,u^3. \tag{4.3}$$

Remark 10.6. One can use also the second-order formal Lagrangians, (2.39) or (2.40). Since they involve the derivatives of v, one should use Eq. (3.30) for computing the conserved vectors (see Remark 10.5). Since the formal Lagrangians (2.39) and (2.40) contain only the first-order derivatives of v, one can use the operator (3.25)in the truncated form:

$$\widetilde{\mathcal{N}}^{\,i} = \xi^i + W\left[\frac{\partial}{\partial u_i} - D_j\,\frac{\partial}{\partial u_{ij}}\right] + D_j(W)\,\frac{\partial}{\partial u_{ij}} + W_*\,\frac{\partial}{\partial v_i}\,\cdot \tag{4.4}$$

The reckoning shows that the extension (3.4) of X_1 to v coincides with X_1. To find the extension of X_2, we prolong it to the derivatives involved in the KdV equation:

$$X_2 = 2u\frac{\partial}{\partial u} - 3t\frac{\partial}{\partial t} - x\frac{\partial}{\partial x} + 5u_t\frac{\partial}{\partial u_t} + 3u_x\frac{\partial}{\partial u_x} + 4u_{xx}\frac{\partial}{\partial u_{xx}} + 5u_{xxx}\frac{\partial}{\partial u_{xxx}}\,,$$

and get

$$X_2(u_t - uu_x - u_{xxx}) = 5(u_t - uu_x - u_{xxx}).$$

Whence $\lambda = 5$. Since

$$D_i(\xi^i) = -4,$$

Eq. (3.8) yields

$$\eta_* = -v.$$

Thus, the extension (3.4) of X_2 is

$$Y_2 = 2u\frac{\partial}{\partial u} - 3t\frac{\partial}{\partial t} - x\frac{\partial}{\partial x} - v\frac{\partial}{\partial v}\,\cdot$$

Eqs. (3.26) yield

$$W = 1 + tu_x, \quad W_* = tv_x$$

and

$$W = 2u + 3tu_t + xu_x, \quad W_* = -v + 3tv_t + xv_x$$

for $Y_1 = X_1$ and Y_2, respectively. Substituting these expressions for W, W_* in (4.4) and applying Eq. (3.30), e.g. to the formal Lagrangian (2.39), we arrive again the conserved vectors (4.2) and (4.3). It is manifest from these calculations that the use of the second-order formal Lagrangians does not simplify the computation of conserved vectors. Therefore, I will will use further the third-order formal Lagrangian (2.38).

4.2 Local symmetries give nonlocal conservation laws

Let us find conservation laws associated with the known infinite algebra of local (Lie-Bäcklund) and nonlocal symmetries of the KdV equation (see, e.g. [62]; see also [34], Ch. 4 and [89], Ch. 5, and the references therein). I will write here only the first component of the corresponding conserved vector (4.1). Let us begin with local symmetries.

The Lie-Bäcklund symmetry of the lowest (fifth) order is

$$X_3 = f_5\frac{\partial}{\partial u} \quad \text{with} \quad f_5 = u_5 + \frac{5}{3}uu_3 + \frac{10}{3}u_1u_2 + \frac{5}{6}u^2u_1, \tag{4.5}$$

where $u_1 = u_x, \; u_2 = u_{xx}, \ldots$. The reckoning shows that the invariance condition (3.10) for $F = u_t - uu_x - u_{xxx}$ is satisfied in the following form:

$$X_3(F) = \Big[\frac{5}{3}(u_3 + uu_1) + \frac{5}{6}(4u_2 + u^2)D_x + \frac{10}{3}u_1 D_x^2 + \frac{5}{3}uD_x^3 + D_x^5\Big](F).$$

The first component of the nonlocal conserved vector (4.1) is $C^1 = vf_5$, i.e.

$$C^1 = \Big(u_5 + \frac{5}{3}uu_3 + \frac{10}{3}u_1u_2 + \frac{5}{6}u^2u_1\Big)v. \tag{4.6}$$

Upon setting $v = u$, we have

$$\Big(u_5 + \frac{5}{3}uu_3 + \frac{10}{3}u_2u_2 + \frac{5}{6}u^2u_1\Big)u = D_x\big(uu_4 - u_1u_3 + \frac{1}{2}u_2^2 + \frac{5}{3}u^2u_2 + \frac{5}{24}u^4\big).$$

Hence, the Lie-Bäcklund symmetry (4.5) provides only a trivial local conserved vector, i.e. with $C^1 = 0$. The reckoning shows that all local higher-order (Lie-Bäcklund) symmetries lead to nonlocal conserved vectors, the first component of which are similar to (4.6), but contain higher-order derivatives, and vanish upon setting $v = u$. Thus, the *local higher-order symmetries lead to essentially nonlocal conservation laws.*

4.3 Nonlocal symmetries give local conservation laws

Let us apply our technique to nonlocal symmetries (see Theorem 10.6). The KdV equation has an infinite set of nonlocal symmetries, namely:

$$\mathcal{X}_{n+2} = g_{n+2}\frac{\partial}{\partial u}\,, \tag{4.7}$$

where g_{n+2} are given recurrently by ([62], see also [34], Eq. (18.36))

$$g_1 = 1 + tu_1, \quad g_{n+2} = \Big(D_x^2 + \frac{2}{3}\,u + \frac{1}{3}\,u_1 D_x^{-1}\Big)g_n, \quad n = 1, 3, \ldots. \tag{4.8}$$

The operator

$$\mathcal{X}_1 = (1 + tu_1)\frac{\partial}{\partial u}$$

corresponding to g_1 is the canonical Lie-Bäcklund representation of the generator X_1 of the Galilean transformation (Example 10.1). Eq. (4.8) yields

$${}_3 = \frac{1}{3}\,[2u + 3t(u_3 + uu_1) + xu_1] \equiv \frac{1}{3}\,(2u + 3tu_t + xu_x),$$

hence $\mathcal{X}_3$ coincides, up to the constant factor 1/3, with the canonical Lie-Bäcklund representation of the scaling generator X_2 (cf. Example 10.2).

Continuing the recursion (4.8), we arrive at the following nonlocal symmetry of the KdV equation:

$$\mathcal{X}_5 = g_5 \frac{\partial}{\partial u} \quad \text{with} \quad g_5 = tf_5 + \frac{x}{3}(u_3 + uu_1) + \frac{4}{3}u_2 + \frac{4}{9}u^2 + \frac{1}{9}u_1\varphi,$$

where f_5 is the coordinate of the Lie-Bäcklund operator X_3 used above and φ is a *nonlocal variable* defined by the following integrable system of equations:

$$\varphi_x = u, \quad \varphi_t = u_{xx} + \frac{1}{2}u^2 \cdot$$

The first component of the conserved vector (4.1) is given by $C^1 = vg_5$. Setting $v = u$, transferring the terms of the form $D_x(\ldots)$ from C^1 to C^2, eliminating an immaterial constant factor and returning to the original notation $u_1 = u_x$, we arrive at a non-trivial conservation law with

$$C^1 = u^3 - 3u_x^2 \tag{4.9}$$

not containing the nonlocal variable φ. We can also take in $C^1 = vg_5$ the solution $v = 1$ of the adjoint equation (2.37), $v_t = v_{xxx} + uv_x$. Then we will arrive again at the conserved vector (4.3).

Dealing likewise with all nonlocal symmetries (4.7), we obtain the renown infinite set of non-trivial conservation laws of the KdV equation. For example, $\mathcal{X}_7$ yields

$$C^1 = 29\, u^4 + 852\, uu_1^2 - 252\, u_2^2. \tag{4.10}$$

5 Application to the Black-Scholes equation

Consider the Black-Scholes equation

$$u_t + \frac{1}{2}A^2x^2u_{xx} + Bxu_x - Cu = 0, \quad A,\ B,\ C = \text{const}. \tag{5.1}$$

Its adjoint equation is

$$\frac{1}{2}A^2x^2v_{xx} + (2A^2 - B)xv_x - v_t + (A^2 - B - C)v = 0. \tag{5.2}$$

Eq. (5.1) does not have the usual Lagrangian. Therefore we will use its formal Lagrangian

$$\mathcal{L} = (u_t + \frac{1}{2}A^2x^2u_{xx} + Bxu_x - Cu)v. \tag{5.3}$$

Let us find the nonlocal conservation law corresponding to the time-translational invariance of Eq. (5.1), i.e. provided by the infinitesimal symmetry

$$X = \frac{\partial}{\partial t} \cdot$$

For this operator, we have $W = -u_t$, and Eq. (1.7) written for the second-order Lagrangian (5.3) yields the conserved vector

$$\begin{aligned} C^1 &= \mathcal{L} - u_t\, v = \Big(\frac{1}{2}A^2x^2u_{xx} + Bxu_x - Cu\Big)v, \\ C^2 &= -\Big[Bxv - D_x\Big(\frac{1}{2}A^2x^2v\Big)\Big]u_t - \frac{1}{2}A^2x^2vu_{tx} \\ &= \Big[-Bxv + A^2xv + \frac{1}{2}A^2x^2v_x\Big]u_t - \frac{1}{2}A^2x^2vu_{tx}. \end{aligned}$$

We can substitute here any solution $v = v(t, x)$ of the adjoint equation (5.2). Let us take, e.g. the invariant solution obtained by using the dilation generator

$$X = x\frac{\partial}{\partial x} - v\frac{\partial}{\partial v} \cdot$$

The reckoning shows that the invariant solution has the form

$$v = \frac{1}{x}\,\mathrm{e}^{-Ct},$$

where C is taken from Eq. (5.1). Substituting the solution v it in the above C^1, C^2 and simplifying as before we obtain the local conserved vector

$$C^1 = \frac{u}{x}\,\mathrm{e}^{-Ct}, \quad C^2 = \Big(\frac{1}{2}A^2xu_x + Bu - \frac{1}{2}A^2u\Big)\,\mathrm{e}^{-Ct}. \tag{5.4}$$

6 Further discussion

6.1 Local and nonlocal conservation laws for not self-adjoint nonlinear equations having Lagrangian

Noether's theorem on local conservation laws (Theorem 10.1) and the theorem on nonlocal conservation laws (Theorem 10.7) are distinctly different even for equations having usual Lagrangians. To illustrate the difference, consider the following examples.

Example 10.13. Consider the equation

$$2u_{tx} + u_x u_{xx} - u_{yy} = 0, \tag{6.1}$$

describing the non-steady-state potential gas flow with transonic speeds. It has the Lagrangian

$$\mathcal{L} = -u_t u_x - \frac{1}{6}u_x^3 + \frac{1}{2}u_y^2. \tag{6.2}$$

Application of Theorem 10.1, e.g. to the generator

$$X = \frac{\partial}{\partial t} \tag{6.3}$$

of the time translation group leaving invariant the variational integral with the Lagrangian (6.2) provides the *local conservation law*

$$D_t(C^1) + D_x(C^2) + D_y(C^3) = 0$$

with the following components*:

$$C^1 = \frac{1}{2}\,u_y^2 - \frac{1}{6}\,u_x^3, \quad C^2 = u_t^2 + \frac{1}{2}\,u_t u_x^2, \quad C^3 = -u_t u_y. \tag{6.4}$$

On the other hand, Eq. (6.1) has the second-order formal Lagrangian

$$\mathcal{L}_* = (2u_{tx} + u_x u_{xx} - u_{yy})v \tag{6.5}$$

which can be replaced by the first-order formal Lagrangian (cf. (6.2))

$$\mathcal{L}_* = -2v_x u_t - \frac{1}{2}v_x u_x^2 + v_y u_y. \tag{6.6}$$

Accordingly, the adjoint equation to (6.1) is

$$2v_{tx} + u_x v_{xx} + v_x u_{xx} - v_{yy} = 0. \tag{6.7}$$

It is manifest from Eqs. (6.1) and (6.7) that Eq. (6.1) is not self-adjoint.

Application of Theorem 10.7 to any of the equivalent formal Lagrangians (6.5) and (6.6) furnishes the following *nonlocal conserved vector* associated with the time-translation symmetry (6.3):

$$\begin{aligned} C^1 &= v_y u_y - \frac{1}{2}\,v_x u_x^2, \\ C^2 &= 2v_t u_t + u_t u_x v_x + v_t u_x^2, \\ C^3 &= -u_t v_y - v_t u_y. \end{aligned} \tag{6.8}$$

Thus, one symmetry (6.3) generates two different conserved vectors, (6.4) and (6.8).

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*For the calculations see [34], Section 23.3, p. 329. Note that the second component is misprinted there as $C^2 = u_t^2 + \frac{1}{2}\,u_x^2$.

Example 10.14. The nonlinear wave equation

$$u_{tt} - \Delta u + au^3 = 0, \quad a = \text{const.}, \tag{6.9}$$

where Δu is the three-dimensional Laplacian, has the Lagrangian

$$L = |\nabla u|^2 - u_t^2 + \frac{a}{2}u^4. \tag{6.10}$$

Let us write conservation laws in the form

$$D_t(\tau) + (\nabla \cdot \boldsymbol{\chi}) = 0,$$

where τ is the density of the conservation law and $\boldsymbol{\chi} = (\chi^1, \chi^2, \chi^3)$.

Application of Theorem 10.1 to the Lagrangian (6.10) and to the operator (6.3) admitted by Eq. (6.9) yields the conservation law with the density (see [34], Eq. (24.10))

$$\tau = u_t^2 + |\nabla u|^2 + \frac{a}{2}u^4. \tag{6.11}$$

On the other hand, Eq. (6.9) has the second-order formal Lagrangian

$$\mathcal{L}_* = (u_{tt} - \Delta u + au^3)v \tag{6.12}$$

which can be replaced by the first-order formal Lagrangian (cf. (6.10))

$$\mathcal{L}_* = -u_t v_t + \nabla u \cdot \nabla v + 3avu^3. \tag{6.13}$$

Accordingly, the adjoint equation to (6.9) is

$$v_{tt} - \Delta v + 3avu^2 = 0. \tag{6.14}$$

It is manifest from Eqs. (6.9) and (6.14) that Eq. (6.9) is not self-adjoint.

Application of Theorem 10.7 to any of the equivalent formal Lagrangians nonself.eq10 or nonself.eq11 and to the symmetry (6.3) yields the *nonlocal conservation law* with the density (cf. (6.11))

$$\tau = u_t v_t + |\nabla u| \cdot |\nabla v| + avu^3. \tag{6.15}$$

6.2 Determination of self-adjoint equations

Example 10.15. We have used the remarkable property of the KdV equation to be self-adjoint for deriving an infinite series of local conservation laws. Let us consider a more general set of equations containing the KdV equation as a particular case, namely:

$$u_t - u_{xxx} - f(x, u, u_x) = 0 \tag{6.16}$$

and single out all self-adjoint equations. We have:

$$\frac{\delta}{\delta u}\Big[(u_t - u_{xxx} - f)v\Big] = -v_t + v_{xxx} - vf_u + D_x\big(vf_{u_x}\big).$$

Hence, the adjoint equation to (6.16) has the form

$$-v_t + v_{xxx} + v_x f_{u_x} + \big(-f_u + f_{xu_x} + u_x f_{uu_x} + u_{xx} f_{u_x u_x}\big)v = 0.$$

Letting $v = u$ we obtain

$$-u_t + u_{xxx} + u_x f_{u_x} + \big(-f_u + f_{xu_x} + u_x f_{uu_x} + u_{xx} f_{u_x u_x}\big)u = 0. \tag{6.17}$$

Comparison with (6.16) yields

$$f_{u_x u_x} = 0,$$

whence

$$f(x, u, u_x) = \varphi(x, u)u_x + \psi(x, u). \tag{6.18}$$

Now Eqs. (6.16) and (6.17) take the form

$$u_t - u_{xxx} - \varphi u_x - \psi = 0 \tag{6.19}$$

and

$$u_t - u_{xxx} - \varphi u_x - (\varphi_x - \psi_u)u = 0, \tag{6.20}$$

respectively. Whence, $(\varphi_x - \psi_u)u = \psi$, or

$$u\frac{\partial \psi}{\partial u} + \psi = u\varphi_x. \tag{6.21}$$

Given an arbitrary function $\varphi(x, u)$, we integrate the linear first-order ordinary differential equation (6.21) for ψ with respect to the variable u and obtain

$$\psi(x, u) = \frac{1}{u}\Big[\int u\varphi_x du + \alpha(x)\Big],$$

thus arriving at the following result.

Proposition 10.1. The general self-adjoint equation of the form (6.21) is

$$u_t - u_{xxx} - \varphi(x,u)u_x - \frac{1}{u}\Big[\int u\varphi_x(x,u)du + \alpha(x)\Big] = 0, \qquad (6.22)$$

where $\varphi(x,u)$ and $\alpha(x)$ are arbitrary functions. In particular, the equation

$$u_t - u_{xxx} - f(u,u_x) = 0$$

is self-adjoint if and only if it has the form

$$u_t - u_{xxx} - \varphi(u)u_x - \frac{a}{u} = 0, \quad a = \text{const.} \qquad (6.23)$$

According to Proposition 10.1, the KdV equation (2.36) and the modified KdV equation

$$u_t = u_{xxx} + u^2 u_x \qquad (6.24)$$

are self-adjoint. Using this property of the modified KdV equation and the known recursion operator (see, e.g. [34], Eq. (19.50))

$$\mathcal{R} = D_x^2 + \frac{2}{3}\,u^2 + \frac{2}{3}\,u_x D_x^{-1} u \qquad (6.25)$$

for the modified KdV equation (cf. (6.25) and (4.8)), one can apply Theorem 10.7 to Eq. (6.24) and, proceeding as in Section 4, compute local and nonlocal conservation laws for the modified KdV equation.

Example 10.16. Let us single out the self-adjoint equations from the set of the equations

$$u_t = f(u)u_{xxx}. \qquad (6.26)$$

Writing the adjoint equation:

$$\frac{\delta}{\delta u}\Big[(u_t - f(u)u_{xxx})v\Big] = -v_t - f'(u)vu_{xxx} + D_x^3\big(vf(u)\big) = 0$$

and letting $v = u$, we obtain:

$$-u_t + fu_{xxx} + 3(2f' + uf'')u_x u_{xx} + (3f'' + uf''')u_x^3 = 0. \qquad (6.27)$$

Comparison of Eq. (6.27) with (6.26) yields

$$2f' + uf'' = 0, \quad 3f'' + uf''' = 0.$$

Since the second equation is obtained from the first one by differentiation, we integrate the equation

$$2f' + uf'' = 0$$

and obtain:

$$f(u) = \frac{a}{u} + b, \quad a, b = \text{const.}$$

Proposition 10.2. The general self-adjoint equation of the form (6.26) is

$$u_t = \left(\frac{a}{u} + b\right) u_{xxx}. \tag{6.28}$$

Example 10.17. Consider the second-order equations of the form

$$u_t = f(u) u_{xx}. \tag{6.29}$$

Proceeding as in the previous example, one can prove the following statement.

Proposition 10.3. The general self-adjoint equation of the form (6.29) is

$$u_t = \frac{a}{u}\, u_{xx}. \tag{6.30}$$

Paper 11

Conservation laws for over-determined systems: Maxwell equations

Unabridged Preprint [49]

Abstract. The Maxwell equations in vacuum provide an interesting example for discussing conservation laws for over-determined systems. Since this system has more equations than the number of dependent variables, the electric field $\boldsymbol{E}$ and the magnetic field $\boldsymbol{H}$, it cannot have a usual Lagrangian written in terms of $\boldsymbol{E}$ and $\boldsymbol{H}$. The evolutionary part of Maxwell's equations has a Lagrangian written in terms of $\boldsymbol{E}$ and $\boldsymbol{H}$, but admits neither Lorentz nor conformal transformations. Addition of the equations $\nabla \cdot \boldsymbol{E} = 0$, $\nabla \cdot \boldsymbol{H} = 0$ guarantees the Lorentz and conformal invariance, but destroys the Lagrangian. The aim of the present paper is to attain a harmony between these two contradictory properties and to derive the conservation laws related to all symmetries of Maxwell's equations.

Introduction

It was discovered by Lorentz [85] that that the Maxwell equations in vacuum are invariant under the 10-parameter group of isometric motions in the four-dimensional flat space-time known as the Minkowsky space. Accordingly, this group is known as the Lorentz group. Then it was shown by Bateman [5], [6] and Cunningham [16] that the Maxwell equations in vacuum admit a wider group, namely the 15-parameter group of the conformal transformations in the Minkowsky space. It was proved later [27] that the conformal transformations, together with the simultaneous dilations and *duality rotations* of the electric and magnetic fields, as well as the obvious infinite group expressing the linear superposition principle, furnish the maximal local group of Lie point transformations admitted by the Maxwell equations.

Thus, the Maxwell equations in vacuum

$$\begin{aligned} &\nabla \times \boldsymbol{E} + \frac{\partial \boldsymbol{H}}{\partial t} = 0, \quad \nabla \times \boldsymbol{H} - \frac{\partial \boldsymbol{E}}{\partial t} = 0, \\ &\nabla \cdot \boldsymbol{E} = 0, \quad \nabla \cdot \boldsymbol{H} = 0, \end{aligned}$$

admit a 17-dimensional Lie algebra, L_{17}, along with the infinite-dimensional Lie algebra existing for all linear equations due to the linear superposition principle.

Bessel-Hagen [7] applied Noether's theorem to the 15-dimensional Lie algebra L_{15} of the conformal group (L_{15} is a subalgebra of L_{17}) and, using the variational formulation of Maxwell's equations in terms of the 4-potential for the electromagnetic field, derived 15 conservation laws. In this way, he obtained, along with the well-known theorems on conservation of energy, momentum, angular momentum and the relativistic center-of-mass theorem, five new conservation laws. He wrote about the latter: "The future will show if they have any physical significance". To the best of my knowledge, a physical interpretation and utilization of Bessel-Hagen's new conservation laws is still an open question. Meanwhile, Lipkin [84] discovered ten new conservation laws involving first derivatives of the electric and magnetic vectors. It was shown later [102] that Lipkin's conservation laws were associated with translations and Lorentz transformations. For a review of further investigations in this direction, see [36], Section 8.6, and the references therein. See also [21].

The present paper is a continuation of my recent work [46] with an emphasis on applicability of Noether's theorem to overdetermined systems hinted in [30]. Namely, the evolution equations of the system of Maxwell equations has a Lagrangian written in terms of the electric $\boldsymbol{E}$ and magnetic $\boldsymbol{H}$ vectors, but admit neither Lorentz nor conformal transformations. The additional equations $\nabla \cdot \boldsymbol{E} = 0$, $\nabla \cdot \boldsymbol{H} = 0$ guarantee the Lorentz and conformal invariance, but destroy the Lagrangian. The aim of the present paper is to attain a harmony between these two contradictory properties.

1 Generalities

1.1 The Maxwell equations

Consider the Maxwell equations in vacuum:

$$\nabla \times \boldsymbol{E} + \frac{\partial \boldsymbol{H}}{\partial t} = 0, \quad \nabla \times \boldsymbol{H} - \frac{\partial \boldsymbol{E}}{\partial t} = 0, \tag{1.1}$$

$$\nabla \cdot \boldsymbol{E} = 0, \quad \nabla \cdot \boldsymbol{H} = 0. \tag{1.2}$$

We will also use the coordinate notation

$$\boldsymbol{x} = (x, y, z), \quad \boldsymbol{E} = (E^1, E^2, E^3), \quad \boldsymbol{H} = (H^1, H^2, H^3)$$

and write Eqs. (1.1)–(1.2) in the coordinate form as well:

$$\begin{aligned}
&E^3_y - E^2_z + H^1_t = 0, \qquad H^3_y - H^2_z - E^1_t = 0,\\
&E^1_z - E^3_x + H^2_t = 0, \qquad H^1_z - H^3_x - E^2_t = 0,\\
&E^2_x - E^1_y + H^3_t = 0, \qquad H^2_x - H^1_y - E^3_t = 0,
\end{aligned} \tag{1.1$'$}$$

$$E^1_x + E^2_y + E^3_z = 0, \qquad H^1_x + H^2_y + H^3_z = 0. \tag{1.2$'$}$$

1.2 Symmetries

Eqs. (1.1)-(1.2) are invariant under the translations of time t and the position vector $\boldsymbol{x}$ as well as the simultaneous rotations of the vectors $\boldsymbol{x}$, $\boldsymbol{E}$ and $\boldsymbol{H}$ due to the vector formulation of Maxwell's equations. The corresponding generators provide the following seven infinitesimal symmetries:

$$\begin{aligned}
&X_0 = \frac{\partial}{\partial t}, \quad X_1 = \frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial y}, \quad X_3 = \frac{\partial}{\partial z},\\
&X_{12} = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} + E^2\frac{\partial}{\partial E^1} - E^1\frac{\partial}{\partial E^2} + H^2\frac{\partial}{\partial H^1} - H^1\frac{\partial}{\partial H^2},\\
&X_{13} = z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z} + E^3\frac{\partial}{\partial E^1} - E^1\frac{\partial}{\partial E^3} + H^3\frac{\partial}{\partial H^1} - H^1\frac{\partial}{\partial H^3},\\
&X_{23} = z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z} + E^3\frac{\partial}{\partial E^2} - E^2\frac{\partial}{\partial E^3} + H^3\frac{\partial}{\partial H^2} - H^2\frac{\partial}{\partial H^3}\,.
\end{aligned} \tag{1.3}$$

Besides, the Maxwell equations admit the Lorentz transformations (hyperbolic rotations) with the generators

$$
\begin{aligned}
X_{01} &= t\frac{\partial}{\partial x} + x\frac{\partial}{\partial t} + E^2\frac{\partial}{\partial H^3} + H^3\frac{\partial}{\partial E^2} - E^3\frac{\partial}{\partial H^2} - H^2\frac{\partial}{\partial E^3}\,,\\
X_{02} &= t\frac{\partial}{\partial y} + y\frac{\partial}{\partial t} + E^3\frac{\partial}{\partial H^1} + H^1\frac{\partial}{\partial E^3} - E^1\frac{\partial}{\partial H^3} - H^3\frac{\partial}{\partial E^1}\,,\\
X_{03} &= t\frac{\partial}{\partial z} + z\frac{\partial}{\partial t} + E^1\frac{\partial}{\partial H^2} + H^2\frac{\partial}{\partial E^1} - E^2\frac{\partial}{\partial H^1} - H^1\frac{\partial}{\partial E^2}\,,
\end{aligned}
\tag{1.4}
$$

and the *duality rotations*

$$
\overline{\boldsymbol{E}} = \boldsymbol{E}\cos\alpha - \boldsymbol{H}\sin\alpha, \quad \overline{\boldsymbol{H}} = \boldsymbol{H}\cos\alpha + \boldsymbol{E}\sin\alpha
$$

with the generator

$$
Z_0 = \boldsymbol{E}\cdot\frac{\partial}{\partial\boldsymbol{H}} - \boldsymbol{H}\cdot\frac{\partial}{\partial\boldsymbol{E}} \equiv \sum_{k=1}^{3}\left(E^k\frac{\partial}{\partial H^k} - H^k\frac{\partial}{\partial E^k}\right)\cdot \tag{1.5}
$$

Furthermore, Eqs. (1.1)-(1.2) admit, due to their homogeneity and linearity, the dilation generators

$$
Z_1 = \boldsymbol{E}\cdot\frac{\partial}{\partial\boldsymbol{E}} + \boldsymbol{H}\cdot\frac{\partial}{\partial\boldsymbol{H}} \equiv \sum_{k=1}^{3}E^k\frac{\partial}{\partial E^k} + \sum_{k=1}^{3}H^k\frac{\partial}{\partial H^k} \tag{1.6}
$$

and

$$
Z_2 = t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}\,, \tag{1.7}
$$

and the superposition generator

$$
S = \boldsymbol{E}_*(\boldsymbol{x},t)\cdot\frac{\partial}{\partial\boldsymbol{E}} + \boldsymbol{H}_*(\boldsymbol{x},t)\cdot\frac{\partial}{\partial\boldsymbol{H}}\,, \tag{1.8}
$$

where the vectors

$$
\boldsymbol{E} = \boldsymbol{E}_*(\boldsymbol{x},t),\ \ \boldsymbol{H} = \boldsymbol{H}_*(\boldsymbol{x},t)
$$

solve Eqs. (1.1)-(1.2).

Finally, Eqs. (1.1)-(1.2) admit the conformal transformations with the generators

$$\begin{aligned}
Y_1 &= \left(x^2 - y^2 - z^2 + t^2\right)\frac{\partial}{\partial x} + 2xy\frac{\partial}{\partial y} + 2xz\frac{\partial}{\partial z} + 2xt\frac{\partial}{\partial t}\\
&- \left(4xE^1 + 2yE^2 + 2zE^3\right)\frac{\partial}{\partial E^1} - \left(4xH^1 + 2yH^2 + 2zH^3\right)\frac{\partial}{\partial H^1}\\
&- \left(4xE^2 - 2yE^1 - 2tH^3\right)\frac{\partial}{\partial E^2} - \left(4xH^2 - 2yH^1 + 2tE^3\right)\frac{\partial}{\partial H^2}\\
&- \left(4xE^3 - 2zE^1 + 2tH^2\right)\frac{\partial}{\partial E^3} - \left(4xH^3 - 2zH^1 - 2tE^2\right)\frac{\partial}{\partial H^3}\,,\\
Y_2 &= 2xy\frac{\partial}{\partial x} + \left(y^2 - x^2 - z^2 + t^2\right)\frac{\partial}{\partial y} + 2yz\frac{\partial}{\partial z} + 2yt\frac{\partial}{\partial t}\\
&- \left(4yE^1 - 2xE^2 - 2tH^3\right)\frac{\partial}{\partial E^1} - \left(4yH^1 - 2xH^2 + 2tE^3\right)\frac{\partial}{\partial H^1}\\
&- \left(4yE^2 + 2xE^1 + 2zE^3\right)\frac{\partial}{\partial E^2} - \left(4yH^2 + 2xH^1 + 2zH^3\right)\frac{\partial}{\partial H^2}\\
&- \left(4yE^3 - 2zE^2 + 2tH^1\right)\frac{\partial}{\partial E^3} - \left(4yH^3 - 2zH^2 - 2tE^1\right)\frac{\partial}{\partial H^3}\,,\\
Y_3 &= 2xz\frac{\partial}{\partial x} + 2yz\frac{\partial}{\partial y} + \left(z^2 - x^2 - y^2 + t^2\right)\frac{\partial}{\partial z} + 2zt\frac{\partial}{\partial t} \qquad (1.9)\\
&- \left(4zE^1 - 2xE^3 + 2tH^2\right)\frac{\partial}{\partial E^1} - \left(4zH^1 - 2xH^3 - 2tE^2\right)\frac{\partial}{\partial H^1}\\
&- \left(4zE^2 - 2yE^3 - 2tH^1\right)\frac{\partial}{\partial E^2} - \left(4zH^2 - 2yH^3 + 2tE^1\right)\frac{\partial}{\partial H^2}\\
&- \left(4zE^3 + 2yE^2 + 2xE^1\right)\frac{\partial}{\partial E^3} - \left(4zH^3 + 2yH^2 + 2xH^1\right)\frac{\partial}{\partial H^3}\,,\\
Y_4 &= 2tx\frac{\partial}{\partial x} + 2ty\frac{\partial}{\partial y} + 2tz\frac{\partial}{\partial z} + \left(x^2 + y^2 + z^2 + t^2\right)\frac{\partial}{\partial t}\\
&- \left(4tE^1 + 2yH^3 - 2zH^2\right)\frac{\partial}{\partial E^1} - \left(4tH^1 - 2yE^3 + 2zE^2\right)\frac{\partial}{\partial H^1}\\
&- \left(4tE^2 + 2zH^1 - 2xH^3\right)\frac{\partial}{\partial E^2} - \left(4tH^2 - 2zE^1 + 2xE^3\right)\frac{\partial}{\partial H^2}\\
&- \left(4tE^3 - 2yH^1 + 2xH^2\right)\frac{\partial}{\partial E^3} - \left(4tH^3 + 2yE^1 - 2xE^2\right)\frac{\partial}{\partial H^3}\,.
\end{aligned}$$

It has been proved in [27] that the operators (1.3)–(1.9) span the Lie algebra of the maximal continuous point transformation group admitted by Eqs. (1.1)–(1.2). This algebra comprises the 17-dimensional subalgebra spanned by the operators (1.3)-(1.7) and (1.9), and the infinite-dimensional ideal consisting of the operators (1.8).

1.3 Conservation equations

Conservation laws for the Maxwell equations are written in the differential form as

$$\big(D_t(\tau) + \operatorname{div}\boldsymbol{\chi}\big)\big|_{(1.1)-(1.2)} = 0, \tag{1.10}$$

where τ and $\boldsymbol{\chi} = (\chi^1, \chi^2, \chi^3)$ are termed the *density* of the conservation law and the *flux*, respectively. The divergence $\operatorname{div}\boldsymbol{\chi}$ of the flux vector is given by

$$\operatorname{div}\boldsymbol{\chi} \equiv \nabla\cdot\boldsymbol{\chi} = D_x(\chi^1) + D_y(\chi^2) + D_z(\chi^3). \tag{1.11}$$

The symbol $\big|_{(1.1)-(1.1)}$ means that Eq. (1.10) is satisfied on the solutions of the Maxwell equations (1.1)-(1.2). The density τ and flux $\boldsymbol{\chi}$ may depend, in general, on the variables $t, \boldsymbol{x}$ as well as on the electric and magnetic fields $\boldsymbol{E}, \boldsymbol{H}$ and their derivatives.

It is assumed in what follows that the time derivatives of $\boldsymbol{E}, \boldsymbol{H}$ have been eliminated by using Eqs. (1.1), and hence τ and χ^1, χ^2, χ^3 depend, in general, on $t, \boldsymbol{x}, \boldsymbol{E}, \boldsymbol{H}, \boldsymbol{E}_x, \boldsymbol{E}_y, \boldsymbol{E}_z, \boldsymbol{H}_x, \boldsymbol{H}_y, \boldsymbol{H}_z, \ldots$. Then Eq. (1.10) is written:

$$\big(D_t(\tau)\big|_{(1.1)} + \operatorname{div}\boldsymbol{\chi}\big)\big|_{(1.2)} = 0, \tag{1.12}$$

where $D_t(\tau)\big|_{(1.1)}$ is obtained from $D_t(\tau)$ by substituting the values of $\boldsymbol{E}_t, \boldsymbol{H}_t$ given by Eqs. (1.1), whereas the sign $\big|_{(1.2)}$ indicates that the terms proportional to $\operatorname{div}\boldsymbol{E}$ and $\operatorname{div}\boldsymbol{H}$ have been eliminated by using Eqs. (1.2).

Often the differential conservation law (1.10) is written, using the divergence theorem, in the integral form:

$$\frac{d}{dt}\int \tau\, dxdydz = 0. \tag{1.13}$$

The integral form of conservation laws is preferable from physical point of view because is has a clear physical interpretation. Namely, Eq. (1.13) means that the integral

$$\frac{d}{dt}\int \tau\, dxdydz \tag{1.14}$$

taken for any solution of Maxwell's equations is constant in time.

1.4 Test for conservation densities of Eqs. (1.1)

Sometimes conservation laws are formulated by stating that the integral (1.14) with a certain τ is constant in time, but the corresponding flux $\boldsymbol{\chi}$ is not given. In this case, one can verify that the conservation equation (1.13) is satisfied by using the *tests for conservation densities* given in this and the next sections. We will need the following property of divergencies.

Lemma 11.1. Let

$$\chi^l = \chi^l(t, \boldsymbol{x}, \boldsymbol{E}, \boldsymbol{H}, \boldsymbol{E}_x, \boldsymbol{E}_y, \boldsymbol{E}_z, \boldsymbol{H}_x, \boldsymbol{H}_y, \boldsymbol{H}_z, \ldots), \quad (l = 1, 2, 3)$$

be any smooth functions. Then the following equations are satisfied:

$$\frac{\delta}{\delta \boldsymbol{E}} (\nabla \cdot \boldsymbol{\chi}) = 0, \quad \frac{\delta}{\delta \boldsymbol{H}} (\nabla \cdot \boldsymbol{\chi}) = 0. \tag{1.15}$$

Here the vector valued Euler-Lagrange operators (variational derivatives) are given by

$$\frac{\delta}{\delta \boldsymbol{E}} = \frac{\partial}{\partial \boldsymbol{E}} - D_i \frac{\partial}{\partial \boldsymbol{E}_i} + D_i D_j \frac{\partial}{\partial \boldsymbol{E}_{ij}} - \cdots, \tag{1.16}$$

$$\frac{\delta}{\delta \boldsymbol{H}} = \frac{\partial}{\partial \boldsymbol{H}} - D_i \frac{\partial}{\partial \boldsymbol{H}_i} + D_i D_j \frac{\partial}{\partial \boldsymbol{H}_{ij}} - \cdots, \tag{1.17}$$

where the indices i, j assume the values 1, 2, 3, and the following notation is used:

$$\begin{aligned}
&D_1 = D_x, \quad D_2 = D_y, \quad D_3 = D_z,\\
&\boldsymbol{E}_1 = \boldsymbol{E}_x, \quad \boldsymbol{E}_2 = \boldsymbol{E}_y, \quad \boldsymbol{E}_3 = \boldsymbol{E}_z,\\
&\boldsymbol{H}_1 = \boldsymbol{H}_x, \quad \boldsymbol{H}_2 = \boldsymbol{H}_y, \quad \boldsymbol{H}_3 = \boldsymbol{H}_z,\\
&\boldsymbol{E}_{11} = \boldsymbol{E}_{xx}, \quad \boldsymbol{E}_{12} = \boldsymbol{E}_{xy}, \ \ldots, \ \boldsymbol{E}_{33} = \boldsymbol{E}_{zz},\\
&\boldsymbol{H}_{11} = \boldsymbol{H}_{xx}, \quad \boldsymbol{H}_{12} = \boldsymbol{H}_{xy}, \ \ldots, \ \boldsymbol{H}_{33} = \boldsymbol{H}_{zz}.
\end{aligned} \tag{1.18}$$

Proof. The Lemma is a particular case of the general result (see [39], Section 8.4.1, Exercise 3; see also [46], p. 747, Lemma 2.3) stating that a function $f(x, u, u_{(1)}), \ldots, u_{(s)})$ is a divergence, i.e.

$$f = \mathrm{div}\phi \equiv \sum_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{D_i}(\phi^{\mathrm{i}}),$$

where $\phi = (\phi^1, \ldots, \phi^{\rm n})$ is an n-dimensional vector with components

$$\phi^1 = \phi^1(x, u, u_{(1)}), \ldots, u_{(s-1)}), \ldots, \phi^n = \phi^n(x, u, u_{(1)}), \ldots, u_{(s-1)}),$$

if and only if

$$\frac{\delta f}{\delta u^\alpha} = 0, \quad \alpha = 1, \ldots, m.$$

Here $x = (x^1, \ldots, x^n)$ are independent variables, $u = (u^1, \ldots, u^m)$ are dependent variables, $u_{(1)}$ is the set of the first-order derivatives u_i^α of u^α with respect to x^i, $u_{(2)}$ is the set of the second-order derivatives, etc.

I give below an independent proof of Eqs. (1.15). Let us write these equations in the expanded form used in (1.11):

$$\frac{\delta}{\delta \boldsymbol{E}}\left[D_x(\chi^1) + D_y(\chi^2) + D_z(\chi^3)\right] = 0,$$

$$\frac{\delta}{\delta \boldsymbol{H}}\left[D_x(\chi^1) + D_y(\chi^2) + D_z(\chi^3)\right] = 0. \qquad (1.15')$$

Due to linearity of the variational derivatives (1.16) -(1.17) and the symmetry of Eqs. (1.15′) with respect to the variables x, y, z, it suffices to prove that

$$\frac{\delta}{\delta \boldsymbol{E}} D_x(\chi^l) = 0, \quad \frac{\delta}{\delta \boldsymbol{H}} D_x(\chi^l) = 0, \quad l = 1, 2, 3. \qquad (1.19)$$

Let us begin with the case when χ^l do not depend on derivatives of $\boldsymbol{E}$ and $\boldsymbol{H}$, i.e.

$$\chi^l = \chi^l(t, \boldsymbol{x}, \boldsymbol{E}, \boldsymbol{H}), \quad l = 1, 2, 3. \qquad (1.20)$$

In this case the variational derivatives (1.16)-(1.17) reduce to

$$\frac{\delta}{\delta \boldsymbol{E}} = \frac{\partial}{\partial \boldsymbol{E}} - D_x \frac{\partial}{\partial \boldsymbol{E}_x} - D_y \frac{\partial}{\partial \boldsymbol{E}_y} - D_z \frac{\partial}{\partial \boldsymbol{E}_z}\,, \qquad (1.21)$$

$$\frac{\delta}{\delta \boldsymbol{H}} = \frac{\partial}{\partial \boldsymbol{H}} - D_x \frac{\partial}{\partial \boldsymbol{H}_x} - D_y \frac{\partial}{\partial \boldsymbol{H}_y} - D_z \frac{\partial}{\partial \boldsymbol{H}_z} \qquad (1.22)$$

and have the components

$$\frac{\delta}{\delta E^k} = \frac{\partial}{\partial E^k} - D_x \frac{\partial}{\partial E^k_x} - D_y \frac{\partial}{\partial E^k_y} - D_z \frac{\partial}{\partial E^k_z}\,,$$

$$\frac{\delta}{\delta H^k} = \frac{\partial}{\partial H^k} - D_x \frac{\partial}{\partial H^k_x} - D_y \frac{\partial}{\partial H^k_y} - D_z \frac{\partial}{\partial H^k_z}\,, \quad k = 1, 2, 3.$$

It is manifest that the total differentiation D_x with respect to x,

$$D_x = \frac{\partial}{\partial x} + \boldsymbol{E}_x \cdot \frac{\partial}{\partial \boldsymbol{E}} + \boldsymbol{H}_x \cdot \frac{\partial}{\partial \boldsymbol{H}}\,,$$

commutes with the partial differentiations with respect to $\boldsymbol{E}$ and $\boldsymbol{H}$:

$$\frac{\partial}{\partial \boldsymbol{E}} D_x - D_x \frac{\partial}{\partial \boldsymbol{E}} = 0, \quad \frac{\partial}{\partial \boldsymbol{H}} D_x - D_x \frac{\partial}{\partial \boldsymbol{H}} = 0. \tag{1.23}$$

The similar equations hold for the total differentiations D_y and D_z with respect to y and z, respectively. The equation

$$D_x(\chi^l) = \frac{\partial \chi^l}{\partial x} + \boldsymbol{E}_x \cdot \frac{\partial \chi^l}{\partial \boldsymbol{E}} + \boldsymbol{H}_x \cdot \frac{\partial \chi^l}{\partial \boldsymbol{H}}$$

yields:

$$\frac{\partial}{\partial \boldsymbol{E}_x} D_x(\chi^l) = \frac{\partial \chi^l}{\partial \boldsymbol{E}}\,, \quad \frac{\partial}{\partial \boldsymbol{E}_y} D_x\big(\chi^l\big) = \frac{\partial}{\partial \boldsymbol{E}_z} D_x\big(\chi^l\big) = 0. \tag{1.24}$$

Using the variational derivative (1.21) and invoking Eqs. (1.24), (1.23) we have:

$$\frac{\delta}{\delta \boldsymbol{E}} D_x(\chi^l) = \frac{\partial}{\partial \boldsymbol{E}} D_x(\chi^l) - D_x \frac{\partial \chi^l}{\partial \boldsymbol{E}} = \left(\frac{\partial}{\partial \boldsymbol{E}} D_x - D_x \frac{\partial}{\partial \boldsymbol{E}}\right) \chi^l = 0.$$

The proof of the second equation (1.19) is similar.

One can use an alternative proof of Eqs. (1.19) in coordinates. Then one has, e.g.

$$\frac{\delta}{\delta E^1} D_x(\chi^l) = \frac{\partial^2 \chi^l}{\partial x \partial E^1} + E^i_x \frac{\partial^2 \chi^l}{\partial E^i \partial E^1} + H^i_x \frac{\partial^2 \chi^l}{\partial H^i \partial E^1} - D_x \left(\frac{\partial \chi^l}{\partial E^1}\right).$$

It follows that

$$\frac{\delta}{\delta E^1} D_x(\chi^l) = 0$$

since

$$D_x \left(\frac{\partial \chi^l}{\partial E^1}\right) = \frac{\partial^2 \chi^l}{\partial x \partial E^1} + E^i_x \frac{\partial^2 \chi^l}{\partial E^i \partial E^1} + H^i_x \frac{\partial^2 \chi^l}{\partial H^i \partial E^1}\,.$$

Consider now the case when τ and χ^l involve the first derivatives of the vectors $\boldsymbol{E}$ and $\boldsymbol{H}$:

$$\tau = \tau(t, \boldsymbol{x}, \boldsymbol{E}, \boldsymbol{H}, \boldsymbol{E}_x, \boldsymbol{E}_y, \boldsymbol{E}_z, \boldsymbol{H}_x, \boldsymbol{H}_y, \boldsymbol{H}_z), \tag{1.25}$$

$$\chi^l = \chi^l(t, \boldsymbol{x}, \boldsymbol{E}, \boldsymbol{H}, \boldsymbol{E}_x, \boldsymbol{E}_y, \boldsymbol{E}_z, \boldsymbol{H}_x, \boldsymbol{H}_y, \boldsymbol{H}_z), \quad l = 1, 2, 3.$$

The equation

$$D_x(\chi^l) = \frac{\partial \chi^l}{\partial x} + \boldsymbol{E}_x \cdot \frac{\partial \chi^l}{\partial \boldsymbol{E}} + \boldsymbol{H}_x \cdot \frac{\partial \chi^l}{\partial \boldsymbol{H}} + \boldsymbol{E}_{xi} \cdot \frac{\partial \chi^l}{\partial \boldsymbol{E}_i} + \boldsymbol{H}_{xi} \cdot \frac{\partial \chi^l}{\partial \boldsymbol{H}_i}\,,$$

yields:

$$\begin{aligned} \frac{\partial}{\partial \boldsymbol{E}_x} D_x(\chi^l) &= \frac{\partial \chi^l}{\partial \boldsymbol{E}} + D_x\left(\frac{\partial \chi^l}{\partial \boldsymbol{E}_x}\right), \\ \frac{\partial}{\partial \boldsymbol{E}_y} D_x(\chi^l) &= D_x\left(\frac{\partial \chi^l}{\partial \boldsymbol{E}_y}\right), \\ \frac{\partial}{\partial \boldsymbol{E}_z} D_x(\chi^l) &= D_x\left(\frac{\partial \chi^l}{\partial \boldsymbol{E}_z}\right). \end{aligned} \tag{1.26}$$

Using the notation (1.18) and the Kronecker symbol δ^i_j, one can write Eqs. (1.26) as

$$\frac{\partial}{\partial \boldsymbol{E}_i} D_x(\chi^l) = \delta^i_1 \frac{\partial \chi^l}{\partial \boldsymbol{E}} + D_x\left(\frac{\partial \chi^l}{\partial \boldsymbol{E}_i}\right). \tag{1.27}$$

Eqs. (1.16), (1.27) yield:

$$\begin{aligned} \frac{\delta}{\delta \boldsymbol{E}} D_x(\chi^l) &= \frac{\partial}{\partial \boldsymbol{E}} D_x(\chi^l) - D_i\left[\delta^i_1 \frac{\partial \chi^l}{\partial \boldsymbol{E}} + D_x\left(\frac{\partial \chi^l}{\partial \boldsymbol{E}_i}\right)\right] + D_x D_i\left(\frac{\partial \chi^l}{\partial \boldsymbol{E}_i}\right) \\ &= \frac{\partial}{\partial \boldsymbol{E}} D_x(\chi^l) - D_x\left(\frac{\partial \chi^l}{\partial \boldsymbol{E}}\right) - D_i D_x\left(\frac{\partial \chi^l}{\partial \boldsymbol{E}_i}\right) + D_x D_i\left(\frac{\partial \chi^l}{\partial \boldsymbol{E}_i}\right). \end{aligned}$$

Using the commutation relations (see the first equation (1.23))

$$D_i D_x = D_x D_i, \quad \frac{\partial}{\partial \boldsymbol{E}} D_x = D_x \frac{\partial}{\partial \boldsymbol{E}}$$

we obtain the first equation (1.19):

$$\frac{\delta}{\delta \boldsymbol{E}} D_x(\chi^l) = 0.$$

The proof of the second equation (1.19) is similar. The general case when χ^l depend on higher derivatives is treated likewise.

We turn now to a characterization of conservation densities. Let us first prove the necessary and sufficient conditions for conservation densities for the equations (1.1) taken alone, without Eqs. (1.2).

Theorem 11.1. A function $\tau(t, \boldsymbol{x}, \boldsymbol{E}, \boldsymbol{H}, \boldsymbol{E}_x, \boldsymbol{H}_x, \ldots)$ is a conservation density for Eqs. (1.1) if and only if it satisfies the equations

$$\frac{\delta}{\delta \boldsymbol{E}}\left[D_t(\tau)\big|_{(1.1)}\right] = 0, \quad \frac{\delta}{\delta \boldsymbol{H}}\left[D_t(\tau)\big|_{(1.1)}\right] = 0. \tag{1.28}$$

Proof. The differential equation defining conservation laws for Eqs. (1.1), considered without Eqs. (1.2), is written (cf. Eqs. (1.12))

$$D_t(\tau)\big|_{(1.1)} + \operatorname{div} \boldsymbol{\chi} = 0,$$

Let us write it, using the notation (1.11), as follows:

$$D_t(\tau)\big|_{(1.1)} + D_x(\chi^1) + D_y(\chi^2) + D_z(\chi^3) = 0. \tag{1.29}$$

Taking the variational derivatives and using Eqs. (1.15) we arrive at Eqs. (1.28).

Example 11.1. Let us test $\tau = \frac{1}{2}\big(|\boldsymbol{E}|^2 + |\boldsymbol{H}|^2\big)$ for conservation density. We have:

$$\begin{aligned} D_t(\tau)\big|_{(1.1)} &= \big(\boldsymbol{E}\cdot \boldsymbol{E}_t + \boldsymbol{H}\cdot \boldsymbol{H}_t\big)\big|_{(1.1)} \\ &= \boldsymbol{E}\cdot(\nabla\times \boldsymbol{H}) - \boldsymbol{H}\cdot(\nabla\times \boldsymbol{E}) \\ &= E^1(H^3_y - H^2_z) + E^2(H^1_z - H^3_x) + E^3(H^2_x - H^1_y) \\ &\quad - H^1(E^3_y - E^2_z) - H^2(E^1_z - E^3_x) - H^3(E^2_x - E^1_y). \end{aligned}$$

Therefore

$$\begin{aligned} \frac{\delta D_t(\tau)\big|_{(1.1)}}{\delta E^1} &= H^3_y - H^2_z + D_z(H^2) - D_y(H^3) = 0, \\ \frac{\delta D_t(\tau)\big|_{(1.1)}}{\delta E^2} &= H^1_z - H^3_x - D_z(H^1) + D_x(H^3) = 0, \\ \frac{\delta D_t(\tau)\big|_{(1.1)}}{\delta E^3} &= H^2_x - H^1_y + D_y(H^1) - D_x(H^2) = 0. \end{aligned}$$

Thus, the first equation (1.28) is satisfied. Verification of the second equation (1.28) is similar. Hence,

$$\tau = \frac{1}{2}\big(|\boldsymbol{E}|^2 + |\boldsymbol{H}|^2\big) \tag{1.30}$$

is a conservation density and therefore the integral

$$\int \frac{1}{2}\big(|\boldsymbol{E}|^2 + |\boldsymbol{H}|^2\big)dxdydz \tag{1.31}$$

is constant in time. It is commonly accepted (after J.C. Maxwell's "A Treatise on Electricity and magnetism", see also [98]) as the *electromagnetic energy* for Eqs. (1.1). The differential equation for conservation of energy has the form (see also Section 3.5)

$$D_t\left(\frac{|\boldsymbol{E}|^2 + |\boldsymbol{H}|^2}{2}\right)\bigg|_{(1.1)} + \operatorname{div}\big(\boldsymbol{E}\times\boldsymbol{H}\big) = 0. \tag{1.32}$$

The reckoning shows that Eq. (1.32) is satisfied identically, without using Eqs. (1.2).

Example 11.2. Let us check if the *Poynting vector* $\boldsymbol{\sigma} = \boldsymbol{E}\times\boldsymbol{H}$ is a conservation density for Eqs. (1.1). Consider, e.g. its first component,

$$\sigma_1 = E^2H^3 - E^3H^2.$$

Its time derivative is (see further Eq. (1.38))

$$D_t(\sigma_1)\big|_{(1.1)} = E^2(E^1_y - E^2_x) + H^3(H^1_z - H^3_x) + E^3(E^1_z - E^3_x) + H^2(H^1_y - H^2_x).$$

Therefore, e.g.

$$\frac{\delta D_t(\sigma_1)\big|_{(1.1)}}{\delta E^1} = -E^2_y - E^3_z \neq 0.$$

Eqs. (1.28) are not satisfied, and hence σ^1 is not a conservation density for Eqs. (1.1) considered without Eqs. (1.2). The same is true for all components of $\boldsymbol{\sigma} = \boldsymbol{E}\times\boldsymbol{H}$. See also further Example 11.3.

1.5 Test for conservation densities of Eqs. (1.1)-(1.2)

The following theorem provides convenient necessary conditions for conservation densities for the Maxwell equations (1.1)-(1.2).

Theorem 11.2. Let $\tau(t, \boldsymbol{x}, \boldsymbol{E}, \boldsymbol{H}, \boldsymbol{E}_t, \boldsymbol{H}_t, \boldsymbol{E}_x, \boldsymbol{H}_x, \ldots)$ be a conservation density for the Maxwell equations (1.1)-(1.2). Then the following equations are satisfied:

$$\frac{\delta}{\delta \boldsymbol{E}}\left[\frac{\delta D_t(\tau)\big|_{(1.1)}}{\delta \boldsymbol{E}}\Bigg|_{(1.2)}\right]=0,\quad \frac{\delta}{\delta \boldsymbol{H}}\left[\frac{\delta D_t(\tau)\big|_{(1.1)}}{\delta \boldsymbol{E}}\Bigg|_{(1.2)}\right]=0,$$

$$\frac{\delta}{\delta \boldsymbol{E}}\left[\frac{\delta D_t(\tau)\big|_{(1.1)}}{\delta \boldsymbol{H}}\Bigg|_{(1.2)}\right]=0,\quad \frac{\delta}{\delta \boldsymbol{H}}\left[\frac{\delta D_t(\tau)\big|_{(1.1)}}{\delta \boldsymbol{H}}\Bigg|_{(1.2)}\right]=0. \tag{1.33}$$

Proof. The conservation equation (1.12) can be equivalently written in the form

$$D_t(\tau)\big|_{(1.1)}+\nabla\cdot\boldsymbol{\chi}=\alpha\,\nabla\cdot\boldsymbol{E}+\beta\,\nabla\cdot\boldsymbol{H} \tag{1.34}$$

with certain coefficients $\alpha=\alpha(t,\boldsymbol{x},\boldsymbol{E},\boldsymbol{H},\ldots)$, $\beta=\beta(t,\boldsymbol{x},\boldsymbol{E},\boldsymbol{H},\ldots)$. Therefore, using Eqs. (1.15) we obtain:

$$\frac{\delta D_t(\tau)\big|_{(1.1)}}{\delta \boldsymbol{E}}=\frac{\delta}{\delta \boldsymbol{E}}\big(\alpha\,\nabla\cdot\boldsymbol{E}+\beta\,\nabla\cdot\boldsymbol{H}\big),$$
$$\frac{\delta D_t(\tau)\big|_{(1.1)}}{\delta \boldsymbol{H}}=\frac{\delta}{\delta \boldsymbol{H}}\big(\alpha\,\nabla\cdot\boldsymbol{E}+\beta\,\nabla\cdot\boldsymbol{H}\big). \tag{1.35}$$

Let us first assume that α and β do not depend upon derivatives of $\boldsymbol{E}$ and $\boldsymbol{H}$:

$$\alpha=\alpha(t,\boldsymbol{x},\boldsymbol{E},\boldsymbol{H}),\quad \beta=\beta(t,\boldsymbol{x},\boldsymbol{E},\boldsymbol{H}),$$

and calculate, e.g.

$$\frac{\delta}{\delta E^1}\big(\alpha\,\nabla\cdot\boldsymbol{E}+\beta\,\nabla\cdot\boldsymbol{H}\big)=\Big(\frac{\partial}{\partial E^1}-D_j\frac{\partial}{\partial E^1_j}\Big)\big(\alpha\,\nabla\cdot\boldsymbol{E}+\beta\,\nabla\cdot\boldsymbol{H}\big).$$

We have:

$$\Big(\frac{\partial}{\partial E^1}-D_j\frac{\partial}{\partial E^1_j}\Big)\big(\alpha\,\nabla\cdot\boldsymbol{E}+\beta\,\nabla\cdot\boldsymbol{H}\big)=\frac{\partial\alpha}{\partial E^1}\,\nabla\cdot\boldsymbol{E}+\frac{\partial\beta}{\partial E^1}\,\nabla\cdot\boldsymbol{H}-D_j(\alpha\delta^j_1),$$

where δ^j_1 is the Kronecker symbol (cf. Eq. (1.27)). Therefore

$$\frac{\delta}{\delta E^1}\Big(\alpha\,\nabla\cdot\boldsymbol{E}+\beta\,\nabla\cdot\boldsymbol{H}\Big)\Big|_{(1.2)}=-D_x(\alpha),$$

and hence, invoking the first equation (1.35):

$$\frac{\delta D_t(\tau)\big|_{(1.1)}}{\delta E^1}\Bigg|_{(1.2)}=-D_x(\alpha).$$

It is manifest now that

$$\frac{\delta D_t(\tau)\big|_{(1.1)}}{\delta E^k}\Bigg|_{(1.2)} = -D_k(\alpha), \tag{1.36}$$

$$\frac{\delta D_t(\tau)\big|_{(1.1)}}{\delta H^k}\Bigg|_{(1.2)} = -D_k(\beta), \quad k = 1,2,3.$$

Therefore Eqs. (1.19) provide the coordinate form of Eqs. (1.33),

$$\frac{\delta}{\delta E^i}\left\{\frac{\delta D_t(\tau)\big|_{(1.1)}}{\delta E^k}\Bigg|_{(1.2)}\right\} = 0, \quad \frac{\delta}{\delta H^i}\left\{\frac{\delta D_t(\tau)\big|_{(1.1)}}{\delta E^k}\Bigg|_{(1.2)}\right\} = 0,$$

$$\frac{\delta}{\delta E^i}\left\{\frac{\delta D_t(\tau)\big|_{(1.1)}}{\delta H^k}\Bigg|_{(1.2)}\right\} = 0, \quad \frac{\delta}{\delta H^i}\left\{\frac{\delta D_t(\tau)\big|_{(1.1)}}{\delta H^k}\Bigg|_{(1.2)}\right\} = 0,$$

where $i, k = 1,2,3$. This completes the proof of the theorem.

Example 11.3. Let us test for conservation density the Poynting vector $\boldsymbol{\sigma} = \boldsymbol{E} \times \boldsymbol{H}$ (cf. Example 11.2). Consider, e.g. its first component,

$$\sigma_1 = E^2H^3 - E^3H^2. \tag{1.37}$$

We have:

$$\begin{aligned} D_t(\sigma_1)\big|_{(1.1)} &= \left(E^2H_t^3 + H^3E_t^2 - E^3H_t^2 - H^2E_t^3\right)\big|_{(1.1)} \\ &= E^2(E_y^1 - E_x^2) + H^3(H_z^1 - H_x^3) \\ &+ E^3(E_z^1 - E_x^3) + H^2(H_y^1 - H_x^2). \end{aligned} \tag{1.38}$$

Hence

$$\frac{\delta D_t(\sigma_1)\big|_{(1.1)}}{\delta E^1} = -E_y^2 - E_z^3, \quad \frac{\delta D_t(\sigma_1)\big|_{(1.1)}}{\delta E^2} = E_y^1, \quad \frac{\delta D_t(\sigma_1)\big|_{(1.1)}}{\delta E^3} = E_z^1,$$

$$\frac{\delta D_t(\sigma_1)\big|_{(1.1)}}{\delta H^1} = -H_y^2 - H_z^3, \quad \frac{\delta D_t(\sigma_1)\big|_{(1.1)}}{\delta H^2} = H_y^1, \quad \frac{\delta D_t(\sigma_1)\big|_{(1.1)}}{\delta H^3} = H_z^1.$$

Therefore Eqs. (1.33) are satisfied, and we have in accordance with Eqs. (1.36):

$$\frac{\delta D_t(\sigma_1)\big|_{(1.1)}}{\delta E^1}\Bigg|_{(1.2)} = E_x^1, \quad \frac{\delta D_t(\sigma_1)\big|_{(1.1)}}{\delta E^2}\Bigg|_{(1.2)} = E_y^1, \quad \frac{\delta D_t(\sigma_1)\big|_{(1.1)}}{\delta E^3}\Bigg|_{(1.2)} = E_z^1,$$

$$\left.\frac{\delta D_t(\sigma_1)\big|_{(1.1)}}{\delta H^1}\right|_{(1.2)} = H_x^1, \quad \left.\frac{\delta D_t(\sigma_1)\big|_{(1.1)}}{\delta H^2}\right|_{(1.2)} = H_y^1, \quad \left.\frac{\delta D_t(\sigma_1)\big|_{(1.1)}}{\delta H^3}\right|_{(1.2)} = H_z^1.$$

Accordingly, σ_1 is the density of the conservation equation of the form (1.34):

$$D_t(\sigma_1)\big|_{(1.1)} + D_x(T_1^1) + D_y(T_1^2) + D_z(T_1^3) = -E^1\,\nabla\cdot\boldsymbol{E} - H^1\,\nabla\cdot\boldsymbol{H}, \quad (1.39)$$

where

$$T_1^1 = \frac{1}{2}\big[(E^2)^2 + (E^3)^2 - (E^1)^2 + (H^2)^2 + (H^3)^2 - (H^1)^2\big],$$
$$T_1^2 = -\big(E^1E^2 + H^1H^2\big), \quad \chi_1^3 = -\big(E^1E^3 + H^1H^3\big),$$

or

$$T_1^i = -E^1E^i - H^1H^i + \frac{1}{2}\delta_1^i\big(|\boldsymbol{E}|^2 + |\boldsymbol{H}|^2\big], \quad i = 1,2,3. \quad (1.40)$$

The obvious generalization of (1.40) is written:

$$T_k^i = -E^kE^i - H^kH^i + \frac{1}{2}\delta_k^i\big(|\boldsymbol{E}|^2 + |\boldsymbol{H}|^2\big), \quad k,i = 1,2,3. \quad (1.41)$$

It is called *Maxwell's tension tensor.*

Eq. (1.39) together with the similar conservation equations associated with the components

$$\sigma_2 = E^3H^1 - E^1H^3, \quad \sigma_3 = E^1H^2 - E^2H^1 \quad (1.42)$$

of the vector $\boldsymbol{\sigma} = \boldsymbol{E}\times\boldsymbol{H}$ provide the following three conservation equations for the Maxwell equations (1.1)-(1.2):

$$D_t(\sigma_k)\big|_{(1.1)} + D_x(T_k^1) + D_y(T_k^2) + D_z(T_k^3) = -E^k\,\nabla\cdot\boldsymbol{E} - H^k\,\nabla\cdot\boldsymbol{H}, \quad (1.43)$$

where $k = 1,2,3$. The quantities σ_k are determined by Eqs. (1.37), (1.42), and T_k^i is Maxwell's tension tensor (1.41).

Thus, the vector $\boldsymbol{\sigma} = \boldsymbol{E}\times\boldsymbol{H}$ is a conservation density for the system of equations (1.1)-(1.2), and hence the integral (known in physics as the *linear momentum*)

$$\int \big(\boldsymbol{E}\times\boldsymbol{H}\big)dxdydz \quad (1.44)$$

is constant in time for the Maxwell equations.

Example 11.4. Let us test for conservation density the vector

$$\boldsymbol{\mu} = \boldsymbol{x} \times (\boldsymbol{E} \times \boldsymbol{H}).$$

Consider, e.g. its first component,

$$\mu^1 = y(E^1H^2 - E^2H^1) - z(E^3H^1 - E^1H^3). \tag{1.45}$$

We have:

$$\begin{aligned}
&D_t(\mu^1)\big|_{(1.1)}\\
&= y\big[H^2(H^3_y - H^2_z) + E^1(E^3_x - E^1_z) + H^1(H^3_x - H^1_z) + E^2(E^3_y - E^2_z)\big]\\
&+ z\big[H^1(H^1_y - H^2_x) + E^3(E^3_y - E^2_z) + H^3(H^3_y - H^2_z) + E^1(E^1_y - E^2_x)\big],
\end{aligned}$$

whence

$$\begin{aligned}
\frac{\delta D_t(\mu^1)\big|_{(1.1)}}{\delta E^1} &= yE^3_x - zE^2_x,\\
\frac{\delta D_t(\mu^1)\big|_{(1.1)}}{\delta E^2} &= yE^3_y + E^3 + zE^3_z + zE^1_x,\\
\frac{\delta D_t(\mu^1)\big|_{(1.1)}}{\delta E^3} &= -yE^1_x - yE^2_y - E^2 - zE^2_z.
\end{aligned}$$

Hence, Eqs. (1.28) are not satisfied. However Eqs. (1.33) are satisfied because, e.g.

$$\begin{aligned}
\left.\frac{\delta D_t(\mu^1)\big|_{(1.1)}}{\delta E^1}\right|_{(1.2)} &= D_x(yE^3 - zE^2),\\
\left.\frac{\delta D_t(\mu^1)\big|_{(1.1)}}{\delta E^2}\right|_{(1.2)} &= yE^3_y + E^3 - zE^2_y = D_y(yE^3 - zE^2),\\
\left.\frac{\delta D_t(\mu^1)\big|_{(1.1)}}{\delta E^3}\right|_{(1.2)} &= yE^3_z - E^2 - zE^2_z = D_z(yE^3 - zE^2).
\end{aligned}$$

Therefore, the integral (known in physics as the *angular momentum*)

$$\int \big[\boldsymbol{x} \times (\boldsymbol{E} \times \boldsymbol{H})\big] dxdydz \tag{1.46}$$

is constant in time for the Maxwell equations (1.1)-(1.2).

2 Evolutionary part of Maxwell's equations

The Maxwell equations in vacuum (1.1)-(1.2) contain six dependent variables, namely, the components of the electric and the magnetic fields,

$$\boldsymbol{E} = (E^1, E^2, E^3), \quad \boldsymbol{H} = (H^1, H^2, H^3),$$

and eight equations. Hence, the system (1.1)-(1.2) is *over-determined* and therefore it does not have a Lagrangian. In this section we will single out the evolutionary part (1.1),

$$\begin{aligned} \nabla \times \boldsymbol{E} + \frac{\partial \boldsymbol{H}}{\partial t} = 0, \\ \nabla \times \boldsymbol{H} - \frac{\partial \boldsymbol{E}}{\partial t} = 0, \end{aligned} \tag{1.1}$$

of Maxwell's equations and discuss the Lagrangian and the symmetries of the system (1.1). Note that If Eqs. (1.2), $\nabla \cdot \boldsymbol{E} = 0$, $\nabla \cdot \boldsymbol{H} = 0$, are satisfied at the initial time $t = 0$, then Eqs. (1.1) guarantee that they hold at any time t. Hence, Eqs. (1.2) are merely initial conditions.

2.1 Lagrangian

Let us test Eqs. (1.1) for self-adjointness. In order to find the adjoint system to (1.1), we introduce six new dependent variables

$$\boldsymbol{V} = (V^1, V^2, V^3), \quad \boldsymbol{W} = (W^1, W^2, W^3)$$

and consider the formal Lagrangian*

$$\mathcal{L} = \boldsymbol{V} \cdot \left(\operatorname{curl} \boldsymbol{E} + \frac{\partial \boldsymbol{H}}{\partial t}\right) + \boldsymbol{W} \cdot \left(\operatorname{curl} \boldsymbol{H} - \frac{\partial \boldsymbol{E}}{\partial t}\right). \tag{2.1}$$

The variational derivatives of the formal Lagrangian (2.1) provide the system (1.1) together with its adjoint, namely:

$$\frac{\delta \mathcal{L}}{\delta \boldsymbol{V}} \equiv \operatorname{curl} \boldsymbol{E} + \frac{\partial \boldsymbol{H}}{\partial t} = 0, \quad \frac{\delta \mathcal{L}}{\delta \boldsymbol{W}} \equiv \operatorname{curl} \boldsymbol{H} - \frac{\partial \boldsymbol{E}}{\partial t} = 0, \tag{2.2}$$

$$\frac{\delta \mathcal{L}}{\delta \boldsymbol{E}} \equiv \operatorname{curl} \boldsymbol{V} + \frac{\partial \boldsymbol{W}}{\partial t} = 0, \quad \frac{\delta \mathcal{L}}{\delta \boldsymbol{H}} \equiv \operatorname{curl} \boldsymbol{W} - \frac{\partial \boldsymbol{V}}{\partial t} = 0. \tag{2.3}$$

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* *Author's note to this 2009 edition:* For definition of formal Lagrangians see Paper 10 in this volume.

If we set $\boldsymbol{V} = \boldsymbol{E}, \boldsymbol{W} = \boldsymbol{H}$, Eqs. (2.3) coincide with (2.2). Hence, the the system (1.1) is self-adjoint. Therefore we set $\boldsymbol{V} = \boldsymbol{E}, \boldsymbol{W} = \boldsymbol{H}$ in (2.1) and obtain the following Lagrangian for Eqs. (1.1) (see [46]):

$$\mathcal{L} = \boldsymbol{E} \cdot \left(\nabla \times \boldsymbol{E} + \frac{\partial \boldsymbol{H}}{\partial t}\right) + \boldsymbol{H} \cdot \left(\nabla \times \boldsymbol{H} - \frac{\partial \boldsymbol{E}}{\partial t}\right). \tag{2.4}$$

For Eqs. (1.1) represented in the coordinate form (1.1′):

$$\begin{aligned} &E_y^3 - E_z^2 + H_t^1 = 0, \qquad H_y^3 - H_z^2 - E_t^1 = 0,\\ &E_z^1 - E_x^3 + H_t^2 = 0, \qquad H_z^1 - H_x^3 - E_t^2 = 0,\\ &E_x^2 - E_y^1 + H_t^3 = 0, \qquad H_x^2 - H_y^1 - E_t^3 = 0, \end{aligned} \tag{1.1$'$}$$

the Lagrangian (2.4) is written:

$$\begin{aligned} \mathcal{L} &= E^1 \,(E_y^3 - E_z^2 + H_t^1) + E^2\,(E_z^1 - E_x^3 + H_t^2) + E^3\,(E_x^2 - E_y^1 + H_t^3)\\ &+ H^1\,(H_y^3 - H_z^2 - E_t^1) + H^2\,(H_z^1 - H_x^3 - E_t^2) + H^3\,(H_x^2 - H_y^1 - E_t^3). \end{aligned}$$

However, by separating Eqs. (1.1) from Eqs. (1.2) we loose certain symmetries of the Maxwell equations. This is discussed in the next section.

Remark 11.1. In the case of the Maxwell equations with electric charges and currents,

$$\nabla \times \boldsymbol{E} + \frac{\partial \boldsymbol{H}}{\partial t} = 0, \quad \nabla \times \boldsymbol{H} - \frac{\partial \boldsymbol{E}}{\partial t} - \boldsymbol{j} = 0, \tag{2.5}$$

$$\nabla \cdot \boldsymbol{E} = \rho, \quad \nabla \cdot \boldsymbol{H} = 0, \tag{2.6}$$

the Lagrangian (2.4) is replaced by*

$$\mathcal{L} = \boldsymbol{E} \cdot \left(\nabla \times \boldsymbol{E} + \frac{\partial \boldsymbol{H}}{\partial t}\right) + \boldsymbol{H} \cdot \left(\nabla \times \boldsymbol{H} - \frac{\partial \boldsymbol{E}}{\partial t} - 2\boldsymbol{j}\right). \tag{2.7}$$

2.2 Symmetries

We know that the Lie algebra L admitted by the Maxwell equations (1.1)–(1.2) is spanned by the operators (1.3)–(1.9).

*Recently I learned that the Lagrangian (2.7) was noted in [2] and in [100]. It was also used in [102]. I thank Professor Bo Thidé for drawing my attention to these papers.

Theorem 11.3. The maximal subalgebra $K \subset L$ admitted by the evolutionary equations (1.1) of Maxwell's system comprises the 10-dimensional algebra spanned by the operators (1.3), (1.5), (1.6), (1.7) and the infinite dimensional ideal (1.8).

Proof. It is geometrically evident that Eqs. (1.1) admit the operators (1.3), (1.6), (1.7) and (1.8). In order to verify that they admit also the duality rotations, let us write the duality generator (1.5) in the prolonged form:

$$Z_0 = \boldsymbol{E}\frac{\partial}{\partial \boldsymbol{H}} - \boldsymbol{H}\frac{\partial}{\partial \boldsymbol{E}} + \boldsymbol{E}_t\frac{\partial}{\partial \boldsymbol{H}_t} - \boldsymbol{H}_t\frac{\partial}{\partial \boldsymbol{E}_t} + \boldsymbol{E}_x\frac{\partial}{\partial \boldsymbol{H}_x} - \boldsymbol{H}_x\frac{\partial}{\partial \boldsymbol{E}_x} + \cdots . \tag{2.8}$$

Acting by the operator (2.8) on the left-hand sides of Eqs. (1.1) we have:

$$\begin{aligned} Z_0\,(\nabla\times\boldsymbol{E} + \boldsymbol{H}_t) &= \boldsymbol{E}_t - \nabla\times\boldsymbol{H},\\ Z_0\big(\nabla\times\boldsymbol{H} - \boldsymbol{E}_t\big) &= \nabla\times\boldsymbol{E} + \boldsymbol{H}_t. \end{aligned} \tag{2.9}$$

Hence, Z_0 is admitted by Eqs. (1.1).

It remains to show that the generators (1.4) of the Lorentz transformations and the generators (1.9) of the conformal transformations are not admitted by Eqs. (1.1) if one takes Eqs. (1.1) alone, without Eqs. (1.2).

Let us begin with the Lorentz transformations. Consider, e.g. the first operator X_{01} from (1.4) and verify that it is not admitted by Eqs. (1.1′). Computing the first prolongation of X_{01} and denoting the prolonged operator again by X_{01} we have:

$$\begin{aligned} X_{01} = {} & t\frac{\partial}{\partial x} + x\frac{\partial}{\partial t} + E^2\frac{\partial}{\partial H^3} + H^3\frac{\partial}{\partial E^2} - E^3\frac{\partial}{\partial H^2} - H^2\frac{\partial}{\partial E^3}\\ & - E^1_t\frac{\partial}{\partial E^1_x} - E^1_x\frac{\partial}{\partial E^1_t} + \big(H^3_x - E^2_t\big)\frac{\partial}{\partial E^2_x} + H^3_y\frac{\partial}{\partial E^2_y} + H^3_z\frac{\partial}{\partial E^2_z}\\ & + \big(H^3_t - E^2_x\big)\frac{\partial}{\partial E^2_t} - \big(H^2_x + E^3_t\big)\frac{\partial}{\partial E^3_x} - H^2_y\frac{\partial}{\partial E^3_y} - H^2_z\frac{\partial}{\partial E^3_z}\\ & - \big(H^2_t + E^3_x\big)\frac{\partial}{\partial E^3_t} - H^1_t\frac{\partial}{\partial H^1_x} - H^1_x\frac{\partial}{\partial H^1_t} - \big(E^3_x + H^2_t\big)\frac{\partial}{\partial H^2_x}\\ & - E^3_y\frac{\partial}{\partial H^2_y} - E^3_z\frac{\partial}{\partial H^2_z} - \big(E^3_t + H^2_x\big)\frac{\partial}{\partial H^2_t} + \big(E^2_x - H^3_t\big)\frac{\partial}{\partial H^3_x}\\ & + E^2_y\frac{\partial}{\partial H^3_y} + E^2_z\frac{\partial}{\partial H^3_z} + \big(E^2_t - H^3_x\big)\frac{\partial}{\partial H^3_t}\,\cdot \end{aligned} \tag{2.10}$$

Acting by the operator (2.10) on the left-hand sides of Eqs. (1.1′) we get:

$$
\begin{aligned}
&X_{01}\big(E_y^3 - E_z^2 + H_t^1\big) = -\big(H_x^1 + H_y^2 + H_z^3\big),\\
&X_{01}\big(E_z^1 - E_x^3 + H_t^2\big) = 0,\\
&X_{01}\big(E_x^2 - E_y^1 + H_t^3\big) = 0,\\
&X_{01}\big(H_y^3 - H_z^2 - E_t^1\big) = E_x^1 + E_y^2 + E_z^3,\\
&X_{01}\big(H_z^1 - H_x^3 - E_t^2\big) = 0,\\
&X_{01}\Big(H_x^2 - H_y^1 - E_t^3\Big) = 0.
\end{aligned} \tag{2.11}
$$

Therefore, Eqs. (1.2′) are required for the invariance of Eqs. (1.1′). Hence, X_{01} is not admitted by Eqs. (1.1). The same is true for all operators (1.4).

Let us apply the similar procedure to the operator Y_1 from (1.9). In order to calculate the action of the prolonged operator Y_1 on the first equation of the system (1.1′), we compute the prolongation to the variables E_y^3, E_z^2, H_t^1 and obtain:

$$
\begin{aligned}
Y_1 = &\big(x^2 - y^2 - z^2 + t^2\big)\frac{\partial}{\partial x} + 2xy\frac{\partial}{\partial y} + 2xz\frac{\partial}{\partial z} + 2xt\frac{\partial}{\partial t}\\
&- \big(4xE^1 + 2yE^2 + 2zE^3\big)\frac{\partial}{\partial E^1} - \big(4xH^1 + 2yH^2 + 2zH^3\big)\frac{\partial}{\partial H^1}\\
&- \big(4xE^2 - 2yE^1 - 2tH^3\big)\frac{\partial}{\partial E^2} - \big(4xH^2 - 2yH^1 + 2tE^3\big)\frac{\partial}{\partial H^2}\\
&- \big(4xE^3 - 2zE^1 + 2tH^2\big)\frac{\partial}{\partial E^3} - \big(4xH^3 - 2zH^1 - 2tE^2\big)\frac{\partial}{\partial H^3}\\
&- \big(6xH_t^1 + 2yH_t^2 + 2zH_t^3 + 2tH_x^1\big)\,\frac{\partial}{\partial H_t^1} - \big(6xE_y^3 - 2yE_x^3\\
&- 2zE_y^1 + 2tH_y^2\big)\frac{\partial}{\partial E_y^3} - \big(6xE_z^2 - 2yE_z^1 - 2zE_x^2 - 2tH_z^3\big)\frac{\partial}{\partial E_z^2}\,\cdot
\end{aligned} \tag{2.12}
$$

Therefore

$$
\begin{aligned}
Y_1\left(E_y^3 - E_z^2 + H_t^1\right) = &- 6x\left(E_y^3 - E_z^2 + H_t^1\right) - 2y\left(E_z^1 - E_x^3 + H_t^2\right)\\
&- 2z\left(E_x^2 - E_y^1 + H_t^3\right) - 2t\left(H_x^1 + H_y^2 + H_z^3\right).
\end{aligned}
$$

Applying the procedure to all equations (1.1′), we obtain the following:

$$
\begin{aligned}
Y_1\left(E_y^3 - E_z^2 + H_t^1\right) &= -\,6x\left(E_y^3 - E_z^2 + H_t^1\right) - 2y\left(E_z^1 - E_x^3 + H_t^2\right)\\
&\quad - 2z\left(E_x^2 - E_y^1 + H_t^3\right) - 2t\left(H_x^1 + H_y^2 + H_z^3\right),\\
Y_1\left(E_z^1 - E_x^3 + H_t^2\right) &= -\,6x\left(E_z^1 - E_x^3 + H_t^2\right) + 2y\left(E_y^3 - E_z^2 + H_t^1\right),\\
Y_1\left(E_x^2 - E_y^1 + H_t^3\right) &= -\,6x\left(E_x^2 - E_y^1 + H_t^3\right) + 2z\left(E_y^3 - E_z^2 + H_t^1\right),\\
Y_1\left(H_y^3 - H_z^2 - E_t^1\right) &= -\,6x\left(H_y^3 - H_z^2 - E_t^1\right) - 2y\left(H_z^1 - H_x^3 - E_t^2\right)\\
&\quad - 2z\left(H_x^2 - H_y^1 - E_t^3\right) + 2t\left(E_x^1 + E_y^2 + E_z^3\right),\\
Y_1\left(H_z^1 - H_x^3 - E_t^2\right) &= -\,6x\left(H_z^1 - H_x^3 - E_t^2\right) + 2y\left(H_y^3 - H_z^2 - E_t^1\right),\\
Y_1\left(H_x^2 - H_y^1 - E_t^3\right) &= -\,6x\left(H_x^2 - H_y^1 - E_t^3\right) + 2z\left(H_y^3 - H_z^2 - E_t^1\right).
\end{aligned}
\tag{2.13}
$$

The first and fourth equations (2.13) show that the infinitesimal tes for the invariance of Eqs. (1.1) under Y_1 requires Eqs. (1.2). Making similar calculations for all operators (1.9), one can verify that the conformal group is not admitted by Eqs. (1.1) taken alone. This completes the proof of the theorem.

Remark 11.2. The complete form of the prolonged operator Y_1 is

$$
\begin{aligned}
Y_1 &= \left(x^2 - y^2 - z^2 + t^2\right)\frac{\partial}{\partial x} + 2xy\frac{\partial}{\partial y} + 2xz\frac{\partial}{\partial z} + 2xt\frac{\partial}{\partial t} + \ldots\\
&- \left(6xE_t^1 + 2yE_t^2 + 2zE_t^3 + 2tE_x^1\right)\frac{\partial}{\partial E_t^1}\\
&- \left(6xH_t^1 + 2yH_t^2 + 2zH_t^3 + 2tH_x^1\right)\frac{\partial}{\partial H_t^1}\\
&- \left(6xE_x^1 + 4E^1 + 2yE_x^2 + 2zE_x^3 + 2yE_y^1 + 2zE_z^1 + 2tE_t^1\right)\frac{\partial}{\partial E_x^1}\\
&- \left(6xH_x^1 + 4H^1 + 2yH_x^2 + 2zH_x^3 + 2yH_y^1 + 2zH_z^1 + 2tH_t^1\right)\frac{\partial}{\partial H_x^1}\\
&- \left(6xE_y^1 + 2E^2 + 2yE_y^2 + 2zE_y^3 - 2yE_x^1\right)\frac{\partial}{\partial E_y^1}\\
&- \left(6xH_y^1 + 2H^2 + 2yH_y^2 + 2zH_y^3 - 2yH_x^1\right)\frac{\partial}{\partial H_y^1}
\end{aligned}
$$

$$- \big(6xE^1_z + 2E^3 + 2yE^2_z + 2zE^3_z - 2zE^1_x\big)\frac{\partial}{\partial E^1_z}$$
$$- \big(6xH^1_z + 2H^3 + 2yH^2_z + 2zH^3_z - 2zH^1_x\big)\frac{\partial}{\partial H^1_z}$$
$$- \big(6xE^2_t - 2H^3 - 2yE^1_t - 2tH^3_t + 2tE^2_x\big)\frac{\partial}{\partial E^2_t}$$
$$- \big(6xH^2_t + 2E^3 - 2yH^1_t + 2tE^3_t + 2tH^2_x\big)\frac{\partial}{\partial H^2_t}$$
$$- \big(6xE^2_x + 4E^2 - 2yE^1_x - 2tH^3_x + 2yE^2_y + 2zE^2_z + 2tE^2_t\big)\frac{\partial}{\partial E^2_x}$$
$$- \big(6xH^2_x + 4H^2 - 2yH^1_x + 2tE^3_x + 2yH^2_y + 2zH^2_z + 2tH^2_t\big)\frac{\partial}{\partial H^2_x}$$
$$- \big(6xE^2_y - 2E^1 - 2yE^1_y - 2tH^3_y - 2yE^2_x\big)\frac{\partial}{\partial E^2_y}$$
$$- \big(6xH^2_y - 2yH^1_y - 2H^1 + 2tE^3_y - 2yH^2_x\big)\frac{\partial}{\partial H^2_y}$$
$$- \big(6xE^2_z - 2yE^1_z - 2zE^2_x - 2tH^3_z\big)\frac{\partial}{\partial E^2_z} - \big(6xH^2_z$$
$$- 2yH^1_z - 2zH^2_x + 2tE^3_z\big)\frac{\partial}{\partial H^2_z}$$
$$- \big(6xE^3_t + 2H^2 - 2zE^1_t + 2tH^2_t + 2tE^3_x\big)\frac{\partial}{\partial E^3_t}$$
$$- \big(6xH^3_t - 2E^2 - 2zH^1_t - 2tE^2_t + 2tH^3_x\big)\frac{\partial}{\partial H^3_t}$$
$$- \big(6xE^3_x + 4E^3 - 2zE^1_x + 2tH^2_x + 2yE^3_y + 2zE^3_z + 2tE^3_t\big)\frac{\partial}{\partial E^3_x}$$
$$- \big(6xH^3_x + 4H^3 - 2zH^1_x - 2tE^2_x + 2yH^3_y + 2zH^3_z + 2tH^3_t\big)\frac{\partial}{\partial H^3_x}$$
$$- \big(6xE^3_y - 2yE^3_x - 2zE^1_y + 2tH^2_y\big)\frac{\partial}{\partial E^3_y}$$
$$- \big(6xH^3_y - 2yH^3_x - 2zH^1_y - 2tE^2_y\big)\frac{\partial}{\partial H^3_y}$$
$$- \big(6xE^3_z - 2E^1 - 2zE^1_z - 2zE^3_x + 2tH^2_x\big)\frac{\partial}{\partial E^3_z}$$
$$- \big(6xH^3_z - 2H^1 - 2zH^1_z - 2zH^3_x - 2tE^2_x\big)\frac{\partial}{\partial H^3_z}\,.$$

Remark 11.3. Acting by the operator (2.10) on the left-hand sides of Eqs. (1.2′) we get:

$$
\begin{aligned}
X_{01}\big(E^1_x + E^2_y + E^3_z\big) &= H^3_y - H^2_z - E^1_t, \\
X_{01}\big(H^1_x + H^2_y + H^3_z\big) &= -\big(E^3_y - E^2_z + H^1_t\big).
\end{aligned}
\tag{2.14}
$$

Eqs. (2.11) and (2.14) show that X_{01} is admitted by the simultaneous system (1.1)–(1.2).

Remark 11.4. Acting by the prolonged operator Y_1 (see Remark 11.2) on Eqs. (1.2′) we have:

$$
\begin{aligned}
Y_1\left(E^1_x + E^2_y + E^3_x\right) &= -6x\left(E^1_x + E^2_y + E^3_x\right) - 2t\left(H^3_y - H^2_z - E^1_t\right), \\
Y_1\left(H^1_x + H^2_y + H^3_x\right) &= -6x\left(H^1_x + H^2_y + H^3_x\right) + 2t\left(E^3_y - E^2_z + H^1_t\right).
\end{aligned}
\tag{2.15}
$$

Eqs. (2.13) and (2.15) show that Y_1 is admitted by the simultaneous system (1.1)–(1.2).

The counterparts of Eqs. (2.13) for the operators Y_2 and Y_3 are obtained from (2.13) by the cyclic permutations of (x, y, z) and the corresponding coordinates of $\boldsymbol{E}$ and $\boldsymbol{H}$.

Remark 11.5. For the operator Y_4 Eqs. (2.13) are replaced by the following equations:

$$
\begin{aligned}
Y_4\left(E^3_y - E^2_z + H^1_t\right) &= -\,6t\left(E^3_y - E^2_z + H^1_t\right) - 2x\left(H^1_x + H^2_y + H^3_z\right), \\
Y_4\left(E^1_z - E^3_x + H^2_t\right) &= -\,6t\left(E^1_z - E^3_x + H^2_t\right) - 2y\left(H^1_x + H^2_y + H^3_z\right), \\
Y_4\left(E^2_x - E^1_y + H^3_t\right) &= -\,6t\left(E^2_x - E^1_y + H^3_t\right) - 2z\left(H^1_x + H^2_y + H^3_z\right), \\
Y_4\left(H^3_y - H^2_z - E^1_t\right) &= -\,6t\left(H^3_y - H^2_z - E^1_t\right) + 2x\left(E^1_x + E^2_y + E^3_z\right), \\
Y_4\left(H^1_z - H^3_x - E^2_t\right) &= -\,6t\left(H^1_z - H^3_x - E^2_t\right) + 2y\left(E^1_x + E^2_y + E^3_z\right), \\
Y_4\left(H^2_x - H^1_y - E^3_t\right) &= -\,6t\left(H^2_x - H^1_y - E^3_t\right) + 2z\left(E^1_x + E^2_y + E^3_z\right).
\end{aligned}
\tag{2.16}
$$

3 Calculation of conservation laws for the evolutionary part of Maxwell's equations

3.1 Notation

I will denote by x^i $(i = 0, 1, 2, 3)$ the independent variables:

$$x^0 = t, \quad x^1 = x, \quad x^2 = y, \quad x^3 = z, \tag{3.1}$$

by u^α $(\alpha = 1, \ldots, 6)$ the dependent variables:

$$u^1 = E^1, \quad u^2 = E^2, \quad u^3 = E^3, \quad u^4 = H^1, \quad u^5 = H^2, \quad u^6 = H^3, \tag{3.2}$$

and write the symmetry generators (1.3), (1.4), (1.6), (1.5) and (1.8) in the form

$$X = \xi^i(x,u)\frac{\partial}{\partial x^i} + \eta^\alpha(x,u)\frac{\partial}{\partial u^\alpha}\cdot \tag{3.3}$$

For the symmetries leaving invariant the variational integral with the Lagrangian $\mathcal{L}$, the conserved vectors $C = (C^0, \ldots, C^3)$ are calculated by

$$C^i = \xi^i\mathcal{L} + \left(\eta^\alpha - \xi^j u_j^\alpha\right)\frac{\partial \mathcal{L}}{\partial u_i^\alpha}\cdot \tag{3.4}$$

Note that the infinitesimal test for the invariance of the variational integral is written

$$X(\mathcal{L}) + \mathcal{L}D_i(\xi^i) = 0, \tag{3.5}$$

where the first prolongation of X is understood. If the invariance condition (3.5) is replaced by the *divergence condition*

$$X(\mathcal{L}) + \mathcal{L}D_i(\xi^i) = D_i(B^i), \tag{3.6}$$

then Eq. (3.4) for the conserved vector is replaced by

$$C^i = \xi^i\mathcal{L} + \left(\eta^\alpha - \xi^j u_j^\alpha\right)\frac{\partial \mathcal{L}}{\partial u_i^\alpha} - B^i. \tag{3.7}$$

Since it suffices to require validity of Eq. (3.5) or Eq. (3.6) on the solutions of Maxwell's equations and since the Lagrangian (2.4) vanishes on the solutions of Eqs. (1.1), one can use Eqs. (3.5), (3.6), (3.4) and (3.7) in the reduced forms

$$X(\mathcal{L}) = 0, \tag{3.8}$$

$$X(\mathcal{L}) = D_i(B^i), \tag{3.9}$$

$$C^i = \big(\eta^\alpha - \xi^j u_j^\alpha\big)\frac{\partial \mathcal{L}}{\partial u_i^\alpha}\,, \quad i = 0, \ldots, 3, \tag{3.10}$$

and

$$C^i = \big(\eta^\alpha - \xi^j u_j^\alpha\big)\frac{\partial \mathcal{L}}{\partial u_i^\alpha} - B^i, \quad i = 0, \ldots, 3, \tag{3.11}$$

respectively. I will denote

$$C^0 = \tau, \quad C^1 = \chi^1, \quad C^2 = \chi^2, \quad C^3 = \chi^3 \tag{3.12}$$

and write the conservation laws in the form (1.12):

$$D_t(\tau) + \operatorname{div}\boldsymbol{\chi} = 0.$$

3.2 Time translation

Let us apply the formula (3.10) to the geometric symmetries (1.3) of Eqs. (1.1). Taking the generator

$$X_0 = \frac{\partial}{\partial t}$$

of time translations we have:

$$\xi^0 = 1, \xi^1 = \xi^2 = \xi^3 = 0, \quad \eta^\alpha = 0, \quad \eta^\alpha - \xi^j u_j^\alpha = -u_t^\alpha.$$

Denoting the density τ of the conservation law provided by X_0 by π_0, we obtain from Eqs. (3.10), using the coordinate form of the Lagrangian (2.4):

$$\pi_0 = -u_t^\alpha \frac{\partial \mathcal{L}}{\partial u_t^\alpha} = -E_t^k \frac{\partial \mathcal{L}}{\partial E_t^k} - H_t^k \frac{\partial \mathcal{L}}{\partial H_t^k} = \sum_{k=1}^{3}\big(H^k E_t^k - E^k H_t^k\big),$$

or

$$\pi_0 = \boldsymbol{H}\cdot\boldsymbol{E}_t - \boldsymbol{E}\cdot\boldsymbol{H}_t. \tag{3.13}$$

The flux components are calculated likewise. Eqs. (3.10), (3.12) and (2.4) yield, e.g.

$$\chi_0^1 = -u_t^\alpha \frac{\partial \mathcal{L}}{\partial u_x^\alpha} = E^2 E_t^3 - E^3 E_t^2 + H^2 H_t^3 - H^3 H_t^2$$

or

$$\chi_0^1 = \big(\boldsymbol{E} \times \boldsymbol{E}_t\big)^1 + \big(\boldsymbol{H} \times \boldsymbol{H}_t\big)^1.$$

Computing the other components of $\boldsymbol{\chi}$ we obtain the flux

$$\boldsymbol{\chi}_0 = \big(\boldsymbol{E} \times \boldsymbol{E}_t\big) + \big(\boldsymbol{H} \times \boldsymbol{H}_t\big). \tag{3.14}$$

Summarizing Equations (1.12), (3.13) and (3.14), we arrive at the following equation:

$$D_t\big(\boldsymbol{H} \cdot \boldsymbol{E}_t - \boldsymbol{E} \cdot \boldsymbol{H}_t\big) + \nabla \cdot \big(\boldsymbol{E} \times \boldsymbol{E}_t + \boldsymbol{H} \times \boldsymbol{H}_t\big) = 0. \tag{3.15}$$

Remark 11.6. Verification of Eq. (3.15) is straightforward. Using Eqs. (1.1) we have:

$$\begin{aligned} D_t\big(\boldsymbol{H} \cdot \boldsymbol{E}_t - \boldsymbol{E} \cdot \boldsymbol{H}_t\big) &= \boldsymbol{H} \cdot \boldsymbol{E}_{tt} - \boldsymbol{E} \cdot \boldsymbol{H}_{tt} \\ &= \boldsymbol{H} \cdot \big(\nabla \times \boldsymbol{H}_t\big) + \boldsymbol{E} \cdot \big(\nabla \times \boldsymbol{E}_t\big) \end{aligned}$$

and

$$\begin{aligned} &\nabla \cdot \big(\boldsymbol{E} \times \boldsymbol{E}_t + \boldsymbol{H} \times \boldsymbol{H}_t\big) = \nabla \cdot \big(\boldsymbol{E} \times \boldsymbol{E}_t\big) + \nabla \cdot \big(\boldsymbol{H} \times \boldsymbol{H}_t\big) \\ &\quad = \boldsymbol{E}_t \cdot \big(\nabla \times \boldsymbol{E}\big) - \boldsymbol{E} \cdot \big(\nabla \times \boldsymbol{E}_t\big) + \boldsymbol{H}_t \cdot \big(\nabla \times \boldsymbol{H}\big) - \boldsymbol{H} \cdot \big(\nabla \times \boldsymbol{H}_t\big). \end{aligned}$$

Therefore

$$\begin{aligned} &D_t\big(\boldsymbol{H} \cdot \boldsymbol{E}_t - \boldsymbol{E} \cdot \boldsymbol{H}_t\big) + \nabla \cdot \big(\boldsymbol{E} \times \boldsymbol{E}_t + \boldsymbol{H} \times \boldsymbol{H}_t\big) \\ &\quad = \boldsymbol{E}_t \cdot \big(\nabla \times \boldsymbol{E}\big) + \boldsymbol{H}_t \cdot \big(\nabla \times \boldsymbol{H}\big) = -\boldsymbol{E}_t \cdot \boldsymbol{H}_t + \boldsymbol{H}_t \cdot \boldsymbol{E}_t = 0. \end{aligned}$$

Representing the differential conservation equation (3.15) in the integral form (1.13), we can formulate the result as follows.

Lemma 11.2. Time translational invariance of Eqs. (1.1) leads to the conservation law

$$\frac{d}{dt}\int \big(\boldsymbol{H} \cdot \boldsymbol{E}_t - \boldsymbol{E} \cdot \boldsymbol{H}_t\big)\, dxdydz = 0. \tag{3.16}$$

Remark 11.7. Eliminating in (3.13) the time-derivatives by using Eqs. (1.1), one has:

$$\pi_0 = \boldsymbol{E} \cdot (\nabla \times \boldsymbol{E}) + \boldsymbol{H} \cdot (\nabla \times \boldsymbol{H}). \tag{3.17}$$

Then the differential conservation law (3.15) is written*

$$D_t\big[\boldsymbol{E} \cdot (\nabla \times \boldsymbol{E}) + \boldsymbol{H} \cdot (\nabla \times \boldsymbol{H})\big] + \operatorname{div}\big[\boldsymbol{E} \times \boldsymbol{E}_t + \boldsymbol{H} \times \boldsymbol{H}_t\big] = 0. \tag{3.18}$$

---

*Eq. (3.18) was discovered by Lipkin [84] (see his Eq. (1)). He expressed his new conservation law in a tensor notation and found nine additional new conservation laws akin to Eq. (3.18). See also [102].

3.3 Spatial translations

For the operator

$$X_1 = \frac{\partial}{\partial x}$$

from (1.3) we have:

$$\xi^0 = 0, \quad \xi^1 = 1, \quad \xi^2 = \xi^3 = 0, \quad \eta^\alpha = 0, \quad \eta^\alpha - \xi^j u_j^\alpha = -u_x^\alpha.$$

Denoting the density τ of the conservation law provided by X_1 by π_1, we obtain from Eqs. (2.4), (3.10):

$$\pi_1 = -u_x^\alpha \frac{\partial \mathcal{L}}{\partial u_t^\alpha} = -E_x^k \frac{\partial \mathcal{L}}{\partial E_t^k} - H_x^k \frac{\partial \mathcal{L}}{\partial H_t^k} = \sum_{k=1}^{3} \big(H^k E_x^k - E^k H_x^k\big).$$

Hence,

$$\pi_1 = \boldsymbol{H} \cdot \boldsymbol{E}_x - \boldsymbol{E} \cdot \boldsymbol{H}_x. \tag{3.19}$$

Let us find the corresponding flux $\boldsymbol{\chi}_1$. Proceeding as above, we have:

$$\begin{aligned}\chi_1^1 = & -u_x^\alpha \frac{\partial \mathcal{L}}{\partial u_x^\alpha} = -E_x^k \frac{\partial \mathcal{L}}{\partial E_x^k} - H_x^k \frac{\partial \mathcal{L}}{\partial H_x^k} \\ & = E^2 E_x^3 - E^3 E_x^2 + H^2 H_x^3 - H^3 H_x^2.\end{aligned}$$

Thus,

$$\chi_1^1 = \big(\boldsymbol{E} \times \boldsymbol{E}_x\big)^1 + \big(\boldsymbol{H} \times \boldsymbol{H}_x\big)^1.$$

Computing the other components of $\boldsymbol{\chi}_1$ we have:

$$\boldsymbol{\chi}_1 = \big(\boldsymbol{E} \times \boldsymbol{E}_x\big) + \big(\boldsymbol{H} \times \boldsymbol{H}_x\big). \tag{3.20}$$

Let us verify that (3.19) and (3.20) satisfy the conservation law (1.12). We have:

$$\begin{aligned} & D_t\big(\boldsymbol{H} \cdot \boldsymbol{E}_x - \boldsymbol{E} \cdot \boldsymbol{H}_x\big) = \boldsymbol{H} \cdot \boldsymbol{E}_{tx} + \boldsymbol{H}_t \cdot \boldsymbol{E}_x - \boldsymbol{E} \cdot \boldsymbol{H}_{tx} - \boldsymbol{E}_t \cdot \boldsymbol{H}_x \qquad (3.21) \\ & = \boldsymbol{H} \cdot \big(\nabla \times \boldsymbol{H}_x\big) - \boldsymbol{E}_x \cdot \big(\nabla \times \boldsymbol{E}\big) + \boldsymbol{E} \cdot \big(\nabla \times \boldsymbol{E}_x\big) - \boldsymbol{H}_x \cdot \big(\nabla \times \boldsymbol{H}\big)\end{aligned}$$

and

$$\begin{aligned} & \nabla \cdot \big(\boldsymbol{E} \times \boldsymbol{E}_x\big) + \nabla \cdot \big(\boldsymbol{H} \times \boldsymbol{H}_x\big) \qquad (3.22) \\ & = \boldsymbol{E}_x \cdot \big(\nabla \times \boldsymbol{E}\big) - \boldsymbol{E} \cdot \big(\nabla \times \boldsymbol{E}_x\big) + \boldsymbol{H}_x \cdot \big(\nabla \times \boldsymbol{H}\big) - \boldsymbol{H} \cdot \big(\nabla \times \boldsymbol{H}_x\big).\end{aligned}$$

It is manifest from Eqs. (3.21), (3.22) that the conservation equation (1.12) is satisfied.

Thus, the invariance under the x-translation group with the generator X_1 provides the following conservation equation (1.12):

$$D_t\big(\boldsymbol{H}\cdot\boldsymbol{E}_x-\boldsymbol{E}\cdot\boldsymbol{H}_x\big)+\nabla\cdot\big(\boldsymbol{E}\times\boldsymbol{E}_x+\boldsymbol{H}\times\boldsymbol{H}_x\big)=0.$$

Applying the above procedure to X_2 and X_3 from (1.3), we arrive at the densities

$$\begin{aligned}\pi_1&=\boldsymbol{H}\cdot\boldsymbol{E}_x-\boldsymbol{E}\cdot\boldsymbol{H}_x,\\ \pi_2&=\boldsymbol{H}\cdot\boldsymbol{E}_y-\boldsymbol{E}\cdot\boldsymbol{H}_y,\\ \pi_3&=\boldsymbol{H}\cdot\boldsymbol{E}_z-\boldsymbol{E}\cdot\boldsymbol{H}_z\end{aligned}\tag{3.23}$$

and the corresponding fluxes

$$\begin{aligned}\boldsymbol{\chi}_1&=\big(\boldsymbol{E}\times\boldsymbol{E}_x\big)+\big(\boldsymbol{H}\times\boldsymbol{H}_x\big),\\ \boldsymbol{\chi}_2&=\big(\boldsymbol{E}\times\boldsymbol{E}_y\big)+\big(\boldsymbol{H}\times\boldsymbol{H}_y\big),\\ \boldsymbol{\chi}_3&=\big(\boldsymbol{E}\times\boldsymbol{E}_z\big)+\big(\boldsymbol{H}\times\boldsymbol{H}_z\big)\end{aligned}\tag{3.24}$$

of the following differential conservation laws:

$$D_t\big(\pi_k\big)+\nabla\cdot\big(\boldsymbol{E}\times\boldsymbol{E}_k+\boldsymbol{H}\times\boldsymbol{H}_k\big)=0,\quad k=1,2,3,\tag{3.25}$$

where

$$\boldsymbol{E}_k=\frac{\partial\boldsymbol{E}}{\partial x^k}\,,\quad \boldsymbol{H}_k=\frac{\partial\boldsymbol{H}}{\partial x^k}\,\cdot$$

Using the integral form (1.13) of conservation laws, I formulate the result as follows.

Lemma 11.3. The invariance of Eqs. (1.1) under the translations with respect to the spatial variables leads to the vector valued conservation law

$$\frac{d}{dt}\int\boldsymbol{\pi}\,dxdydz=0,\tag{3.26}$$

where $\boldsymbol{\pi}=(\pi_1,\pi_2,\pi_3)$ is the vector with the components (3.23).

Remark 11.8. According to Eqs. (3.21), (3.23), we have:

$$\begin{aligned}D_t(\pi_1)&=\boldsymbol{H}\cdot\big(\nabla\times\boldsymbol{H}_x\big)-\boldsymbol{E}_x\cdot\big(\nabla\times\boldsymbol{E}\big)+\boldsymbol{E}\cdot\big(\nabla\times\boldsymbol{E}_x\big)-\boldsymbol{H}_x\cdot\big(\nabla\times\boldsymbol{H}\big),\\ D_t(\pi_2)&=\boldsymbol{H}\cdot\big(\nabla\times\boldsymbol{H}_y\big)-\boldsymbol{E}_y\cdot\big(\nabla\times\boldsymbol{E}\big)+\boldsymbol{E}\cdot\big(\nabla\times\boldsymbol{E}_y\big)-\boldsymbol{H}_y\cdot\big(\nabla\times\boldsymbol{H}\big),\\ D_t(\pi_3)&=\boldsymbol{H}\cdot\big(\nabla\times\boldsymbol{H}_z\big)-\boldsymbol{E}_z\cdot\big(\nabla\times\boldsymbol{E}\big)+\boldsymbol{E}\cdot\big(\nabla\times\boldsymbol{E}_z\big)-\boldsymbol{H}_z\cdot\big(\nabla\times\boldsymbol{H}\big).\end{aligned}$$

3.4 Rotations

Applying the formula (3.10) with $i = 0$ to the rotation generator

$$X_{12} = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} + E^2\frac{\partial}{\partial E^1} - E^1\frac{\partial}{\partial E^2} + H^2\frac{\partial}{\partial H^1} - H^1\frac{\partial}{\partial H^2}$$

from (1.3) we obtain the conservation density

$$\tau_{12} = \big(\eta^\alpha - \xi^j u_j^\alpha\big)\frac{\partial \mathcal{L}}{\partial u_t^\alpha}\,,$$

which upon substitution of the coordinates of X_{12} becomes:

$$\begin{aligned}\tau_{12} = &\big(E^2 + xE_y^1 - yE_x^1\big)\frac{\partial \mathcal{L}}{\partial E_t^1} - \big(E^1 - xE_y^2 + yE_x^2\big)\frac{\partial \mathcal{L}}{\partial E_t^2}\\ &+ \big(xE_y^3 - yE_x^3\big)\frac{\partial \mathcal{L}}{\partial E_t^3} + \big(H^2 + xH_y^1 - yH_x^1\big)\frac{\partial \mathcal{L}}{\partial H_t^1}\\ &- \big(H^1 - xH_y^2 + yH_x^2\big)\frac{\partial \mathcal{L}}{\partial H_t^2} + \big(xH_y^3 - yH_x^3\big)\frac{\partial \mathcal{L}}{\partial H_t^3}\,.\end{aligned}$$

Invoking the coordinate expression for the Lagrangian (2.4), we obtain

$$\begin{aligned}\tau_{12} = &\big(E^1 - xE_y^2 + yE_x^2\big)H^2 - \big(E^2 + xE_y^1 - yE_x^1\big)H^1 - \big(xE_y^3 - yE_x^3\big)H^3\\ &+ \big(H^2 + xH_y^1 - yH_x^1\big)E^1 - \big(H^1 - xH_y^2 + yH_x^2\big)E^2 + \big(xH_y^3 - yH_x^3\big)E^3,\end{aligned}$$

or

$$\tau_{12} = 2\big(E^1H^2 - E^2H^1\big) + y\big(\boldsymbol{H}\cdot\boldsymbol{E}_x - \boldsymbol{E}\cdot\boldsymbol{H}_x\big) - x\big(\boldsymbol{H}\cdot\boldsymbol{E}_y - \boldsymbol{E}\cdot\boldsymbol{H}_y\big).$$

Using the notation (3.23), we write it in the form

$$\tau_{12} = 2\big(E^1H^2 - E^2H^1\big) + y\pi_1 - x\pi_2, \tag{3.27}$$

or in terms of the vector $\boldsymbol{\pi} = (\pi_1, \pi_2, \pi_3)$:

$$\tau_{12} = 2\big(\boldsymbol{E}\times\boldsymbol{H}\big)^3 + \big(\boldsymbol{x}\times\boldsymbol{\pi}\big)^3. \tag{3.28}$$

The components of the flux $\boldsymbol{\chi}_{12} = (\chi^1_{12}, \chi^2_{12}, \chi^3_{12})$ are computed by the formulae

$$
\begin{aligned}
\chi^1_{12} = \big(\eta^\alpha - \xi^j u^\alpha_j\big)\frac{\partial \mathcal{L}}{\partial u^\alpha_x} &= \big(xE^2_y - yE^2_x - E^1\big)\frac{\partial \mathcal{L}}{\partial E^2_x} + \big(xE^3_y - yE^3_x\big)\frac{\partial \mathcal{L}}{\partial E^3_x}\\
&+ \big(xH^2_y - yH^2_x - H^1\big)\frac{\partial \mathcal{L}}{\partial H^2_x} + \big(xH^3_y - yH^3_x\big)\frac{\partial \mathcal{L}}{\partial H^3_x}\,,\\
\chi^2_{12} = \big(\eta^\alpha - \xi^j u^\alpha_j\big)\frac{\partial \mathcal{L}}{\partial u^\alpha_y} &= \big(xE^1_y - yE^1_x + E^2\big)\frac{\partial \mathcal{L}}{\partial E^1_y} + \big(xE^3_y - yE^3_x\big)\frac{\partial \mathcal{L}}{\partial E^3_y}\\
&+ \big(xH^1_y - yH^1_x + H^2\big)\frac{\partial \mathcal{L}}{\partial H^1_y} + \big(xH^3_y - yH^3_x\big)\frac{\partial \mathcal{L}}{\partial H^3_y}\,,\\
\chi^3_{12} = \big(\eta^\alpha - \xi^j u^\alpha_j\big)\frac{\partial \mathcal{L}}{\partial u^\alpha_z} &= \big(xE^1_y - yE^1_x + E^2\big)\frac{\partial \mathcal{L}}{\partial E^1_z} + \big(xE^2_y - yE^2_x - E^1\big)\frac{\partial \mathcal{L}}{\partial E^2_z}\\
&+ \big(xH^1_y - yH^1_x + H^2\big)\frac{\partial \mathcal{L}}{\partial H^1_z} + \big(xH^2_y - yH^2_x - H^1\big)\frac{\partial \mathcal{L}}{\partial H^2_z}\,\cdot
\end{aligned}
$$

Substituting here the expression (2.4) for the Lagrangian $\mathcal{L}$, we obtain the following flux components:

$$
\begin{aligned}
\chi^1_{12} &= \big(xE^2_y - yE^2_x - E^1\big)E^3 - \big(xE^3_y - yE^3_x\big)E^2\\
&\quad + \big(xH^2_y - yH^2_x - H^1\big)H^3 - \big(xH^3_y - yH^3_x\big)H^2\,,\\
\chi^2_{12} &= -\big(xE^1_y - yE^1_x + E^2\big)E^3 + \big(xE^3_y - yE^3_x\big)E^1\\
&\quad - \big(xH^1_y - yH^1_x + H^2\big)H^3 + \big(xH^3_y - yH^3_x\big)H^1\,,\\
\chi^3_{12} &= \big(xE^1_y - yE^1_x + E^2\big)E^2 - \big(xE^2_y - yE^2_x - E^1\big)E^1\\
&\quad + \big(xH^1_y - yH^1_x + H^2\big)H^2 - \big(xH^2_y - yH^2_x - H^1\big)H^1\,\cdot
\end{aligned}
$$

It is useful to rewrite them by collecting the terms with x and y :

$$
\begin{aligned}
\chi^1_{12} = -\big(E^1E^3 + H^1H^3\big) &+ x\big(E^3E^2_y - E^2E^3_y + H^3H^2_y - H^2H^3_y\big)\\
&- y\big(E^3E^2_x - E^2E^3_x + H^3H^2_x - H^2H^3_x\big),\\
\chi^2_{12} = -\big(E^2E^3 + H^2H^3\big) &+ x\big(E^1E^3_y - E^3E^1_y + H^1H^3_y - H^3H^1_y\big)\\
&- y\big(E^1E^3_x - E^3E^1_x + H^1H^3_x - H^3H^1_x\big),
\end{aligned}
$$

$$\chi^3_{12} = \big(E^1\big)^2 + \big(E^2\big)^2 + \big(H^1\big)^2 + \big(H^2\big)^2 \qquad (3.29)$$
$$+ x\big(E^2E^1_y - E^1E^2_y + H^2H^1_y - H^1H^2_y\big)$$
$$- y\big(E^2E^1_x - E^1E^2_x + H^2H^1_x - H^1H^2_x\big).$$

To verify that τ_{12} given by (3.27) is a conservation density, we differentiate (3.27) with respect to t, use Eqs. (1.1′) and Remark 11.8, and obtain:

$$D_t(\tau_{12}) \qquad (3.30)$$
$$= 2\big[E^1\big(E^3_x - E^1_z\big) + H^2\big(H^3_y - H^2_z\big) - E^2\big(E^2_x - E^3_y\big) - H^1\big(H^1_z - H^3_x\big)\big]$$
$$+ y\big[\boldsymbol{H}\cdot\big(\nabla\times\boldsymbol{H}_x\big) - \boldsymbol{E}_x\cdot\big(\nabla\times\boldsymbol{E}\big) + \boldsymbol{E}\cdot\big(\nabla\times\boldsymbol{E}_x\big) - \boldsymbol{H}_x\cdot\big(\nabla\times\boldsymbol{H}\big)\big]$$
$$- x\big[\boldsymbol{H}\cdot\big(\nabla\times\boldsymbol{H}_y\big) - \boldsymbol{E}_y\cdot\big(\nabla\times\boldsymbol{E}\big) + \boldsymbol{E}\cdot\big(\nabla\times\boldsymbol{E}_y\big) - \boldsymbol{H}_y\cdot\big(\nabla\times\boldsymbol{H}\big)\big].$$

Calculating of the divergence of the vector $\boldsymbol{\chi}_{12} = (\chi^1_{12}, \chi^2_{12}, \chi^3_{12})$ with the components (3.29) one can check that $\nabla\cdot\boldsymbol{\chi}_{12}$ is equal to the right-hand side of (3.30) taken with the opposite sign. Hence, (3.27), (3.29) satisfy the conservation equation (1.12).

Proceeding likewise with other rotation generators and using the integral form (1.13) of conservation laws, we arrive at the following result.

Lemma 11.4. The invariance of Eqs. (1.1) under the rotations with the generators X_{12}, X_{13}, X_{23} from (1.3) provides the conservation law

$$\frac{d}{dt}\int\big[2\big(\boldsymbol{E}\times\boldsymbol{H}\big) + \big(\boldsymbol{x}\times\boldsymbol{\pi}\big)\big]\,dxdydz = 0, \qquad (3.31)$$

where $\boldsymbol{\pi} = (\pi_1, \pi_2, \pi_3)$ is the vector with the components (3.23).

3.5 Duality rotations

Consider the duality rotations with the generator (1.5). Acting by the prolonged operator Z_0 given by (2.8) on the Lagrangian (2.4) and using Eqs. (2.9) we have:

$$Z_0(\mathcal{L}) = -\boldsymbol{H}\cdot\big(\nabla\times\boldsymbol{E} + \boldsymbol{H}_t\big) + \boldsymbol{E}\cdot\big(\nabla\times\boldsymbol{H} - \boldsymbol{E}_t\big)$$
$$+ \boldsymbol{E}\cdot\big(-\nabla\times\boldsymbol{H} + \boldsymbol{E}_t\big) + \boldsymbol{H}\cdot\big(\nabla\times\boldsymbol{E} + \boldsymbol{H}_t\big) = 0.$$

Hence, Eq. (3.8) is satisfied, and Eq. (3.10) yields:

$$\tau = \boldsymbol{E}\cdot\frac{\partial\mathcal{L}}{\partial\boldsymbol{H}_t} - \boldsymbol{H}\cdot\frac{\partial\mathcal{L}}{\partial\boldsymbol{E}_t} = \boldsymbol{E}\cdot\boldsymbol{E} - \boldsymbol{H}\cdot(-\boldsymbol{H}) = \boldsymbol{E}\cdot\boldsymbol{E} + \boldsymbol{H}\cdot\boldsymbol{H}.$$

Therefore, τ is identical (up to the inessential factor 1/2) with the energy density (1.30),

$$\tau = \frac{1}{2}\big(|\boldsymbol{E}|^2 + |\boldsymbol{H}|^2\big).$$

The reckoning shows that the flux $\boldsymbol{\chi}$ is the Poynting vector

$$\boldsymbol{\sigma} = (\boldsymbol{E} \times \boldsymbol{H}).$$

Thus, the invariance of Eqs. (1.1) with respect to the duality rotations provides the differential equation (1.32) for the conservation of energy,

$$D_t\left(\frac{|\boldsymbol{E}|^2 + |\boldsymbol{H}|^2}{2}\right)\Bigg|_{(1.1)} + \nabla\cdot\big(\boldsymbol{E}\times\boldsymbol{H}\big) = 0,$$

or Eq. (1.31) in the integral form:

$$\frac{d}{dt}\int \frac{1}{2}\big(|\boldsymbol{E}|^2 + |\boldsymbol{H}|^2\big)dxdydz = 0. \tag{1.31}$$

3.6 Dilations

The invariance test (3.8) is satisfied for the dilation generators Z_1 and Z_2 given by (1.6) and (1.7), respectively. The reckoning shows that for Z_1 Eq. (3.10) yields $\tau = 0, \boldsymbol{\chi} = 0$. Hence, the invariance with respect to the dilations of the dependent variables with the generator Z_1 does not provide a nontrivial conservation law.

Let us consider the dilations of the independent variables with the generator Z_2. For this operator, Eq. (3.10) for $\tau = C^0$ is written

$$\tau = -\big(t\boldsymbol{E}_t + x\boldsymbol{E}_x + y\boldsymbol{E}_y + z\boldsymbol{E}_z\big)\frac{\partial\mathcal{L}}{\partial\boldsymbol{E}_t} - \big(t\boldsymbol{H}_t + x\boldsymbol{H}_x + y\boldsymbol{H}_y + z\boldsymbol{H}_z\big)\frac{\partial\mathcal{L}}{\partial\boldsymbol{H}_t}\cdot$$

It follows:

$$\begin{aligned}\tau &= \big(t\boldsymbol{E}_t + x\boldsymbol{E}_x + y\boldsymbol{E}_y + z\boldsymbol{E}_z\big)\cdot\boldsymbol{H} - \big(t\boldsymbol{H}_t + x\boldsymbol{H}_x + y\boldsymbol{H}_y + z\boldsymbol{H}_z\big)\cdot\boldsymbol{E}\\ &= t\big(\boldsymbol{H}\cdot\boldsymbol{E}_t - \boldsymbol{E}\cdot\boldsymbol{H}_t\big) + x\big(\boldsymbol{H}\cdot\boldsymbol{E}_x - \boldsymbol{E}\cdot\boldsymbol{H}_x\big) + y\big(\boldsymbol{H}\cdot\boldsymbol{E}_y - \boldsymbol{E}\cdot\boldsymbol{H}_y\big)\\ &\quad + z\big(\boldsymbol{H}\cdot\boldsymbol{E}_z - \boldsymbol{E}\cdot\boldsymbol{H}_z\big) = t\pi_0 + x\pi_1 + y\pi_2 + \pi_3,\end{aligned}$$

or

$$\tau = t\pi_0 + \boldsymbol{x}\cdot\boldsymbol{\pi}, \tag{3.32}$$

where π_0 is given by (3.13) or equivalently by (3.17), and $\boldsymbol{\pi} = (\pi_1, \pi_2, \pi_3)$ is the vector with the components (3.23). The flux components are computed likewise, e.g.

$$\begin{aligned}
\chi^1 &= -\big(t\boldsymbol{E}_t + x\boldsymbol{E}_x + y\boldsymbol{E}_y + z\boldsymbol{E}_z\big)\frac{\partial\mathcal{L}}{\partial\boldsymbol{E}_x} - \big(t\boldsymbol{H}_t + x\boldsymbol{H}_x + y\boldsymbol{H}_y + z\boldsymbol{H}_z\big)\frac{\partial\mathcal{L}}{\partial\boldsymbol{H}_x}\\
&= \big(tE_t^3 + xE_x^3 + yE_y^3 + zE_z^3\big)E^2 - \big(tE_t^2 + xE_x^2 + yE_y^2 + zE_z^2\big)E^3\\
&= \big(tH_t^3 + xH_x^3 + yH_y^3 + zH_z^3\big)H^2 - \big(tH_t^2 + xH_x^2 + yH_y^2 + zH_z^2\big)H^3.
\end{aligned}$$

Collecting the like terms, we have:

$$\begin{aligned}
\chi^1 &= t\big(E^2E_t^3 - E^3E_t^2 + H^2H_t^3 - H^3H_t^2\big)\\
&+ x\big(E^2E_x^3 - E^3E_x^2 + H^2H_x^3 - H^3H_x^2\big)\\
&+ y\big(E^2E_y^3 - E^3E_y^2 + H^2H_y^3 - H^3H_y^2\big)\\
&+ z\big(E^2E_z^3 - E^3E_z^2 + H^2H_z^3 - H^3H_z^2\big),
\end{aligned}$$

or, using the vectors $\boldsymbol{\chi}_0 = \big(\chi_0^1, \chi_0^2, \chi_0^3\big)$ and $\boldsymbol{\chi}_i = \big(\chi_i^1, \chi_i^2, \chi_i^3\big)$, $i = 1, 2, 3$, defined by Eqs. (3.14) and (3.24), respectively:

$$\chi^1 = t\chi_0^1 + x\chi_1^1 + y\chi_2^1 + z\chi_3^1.$$

Computing the other flux components, we obtain the following flux:

$$\boldsymbol{\chi} = t\boldsymbol{\chi}_0 + x\boldsymbol{\chi}_1 + y\boldsymbol{\chi}_2 + z\boldsymbol{\chi}_3. \tag{3.33}$$

Thus, we have arrived at the following differential conservation equation:

$$D_t\big(t\pi_0 + \boldsymbol{x}\cdot\boldsymbol{\pi}\big) + \nabla\cdot\big(t\boldsymbol{\chi}_0 + x\boldsymbol{\chi}_1 + y\boldsymbol{\chi}_2 + z\boldsymbol{\chi}_3\big) = 0. \tag{3.34}$$

Representing it in the integral form, we can formulate the result as follows.

Lemma 11.5. The invariance of Eqs. (1.1) with respect to the group of dilations of the independent variables with the generator Z_2 leads to the conservation law

$$\frac{d}{dt}\int\big(t\pi_0 + \boldsymbol{x}\cdot\boldsymbol{\pi}\big)\,dxdydz = 0. \tag{3.35}$$

3.7 Conservation laws provided by superposition

For the superposition generator (1.8),

$$S = \boldsymbol{E}_*(\boldsymbol{x}, t)\cdot\frac{\partial}{\partial \boldsymbol{E}} + \boldsymbol{H}_*(\boldsymbol{x}, t)\cdot\frac{\partial}{\partial \boldsymbol{H}}\,, \tag{1.8}$$

Eq. (3.10) for calculating the conservation density is written

$$\tau = \boldsymbol{E}_*\frac{\partial \mathcal{L}}{\partial \boldsymbol{E}_t} + \boldsymbol{H}_*\frac{\partial \mathcal{L}}{\partial \boldsymbol{H}_t}$$

and yields:

$$\tau = \boldsymbol{H}_*\cdot\boldsymbol{E} - \boldsymbol{E}_*\cdot\boldsymbol{H}. \tag{3.36}$$

The flux components are computed likewise, e.g.

$$\begin{aligned}\chi^1 = E^k_*\frac{\partial \mathcal{L}}{\partial E^k_x} + H^k_*\frac{\partial \mathcal{L}}{\partial H^k_x} &= E^2_*E^3 - E^3_*E^2 + H^2_*H^3 - H^3_*H^2\\ &= \big(\boldsymbol{E}_*\times\boldsymbol{E}\big)^1 + \big(\boldsymbol{H}_*\times\boldsymbol{H}\big)^1.\end{aligned}$$

Hence, computing the other flux components, we obtain the following flux:

$$\boldsymbol{\chi} = \big(\boldsymbol{E}_*\times\boldsymbol{E}\big) + \big(\boldsymbol{H}_*\times\boldsymbol{H}\big). \tag{3.37}$$

Thus, we have arrived at the following differential conservation equation:

$$D_t\big(\boldsymbol{H}_*\cdot\boldsymbol{E} - \boldsymbol{E}_*\cdot\boldsymbol{H}\big) + \nabla\cdot\big(\boldsymbol{E}_*\times\boldsymbol{E} + \boldsymbol{H}_*\times\boldsymbol{H}\big) = 0. \tag{3.38}$$

Representing it in the integral form, we can formulate the result as follows.

Lemma 11.6. The linear superposition principle provides the infinite set of conservation laws*

$$\frac{d}{dt}\int\big(\boldsymbol{H}_*\cdot\boldsymbol{E} - \boldsymbol{E}_*\cdot\boldsymbol{H}\big)\,dxdydz = 0 \tag{3.39}$$

involving any two solutions, $(\boldsymbol{E}, \boldsymbol{H})$ and $(\boldsymbol{E}_*, \boldsymbol{H}_*)$, of Eqs. (1.1)–(1.2).

---

*This conservation law should not be confused with the *Lorentz reciprocity theorem* expressed by the equation $\nabla\cdot\big(\boldsymbol{E}\times\boldsymbol{H}_* - \boldsymbol{E}_*\times\boldsymbol{H}\big) = 0$, see [26], Section 3-8.

Example 11.5. Taking the trivial solution

$$\boldsymbol{E}_* = \text{const.}, \quad \boldsymbol{H}_* = \text{const.}$$

we obtain from (3.39) the conservation equations

$$\frac{d}{dt}\int \boldsymbol{E}\, dxdydz = 0, \quad \frac{d}{dt}\int \boldsymbol{H}\, dxdydz = 0 \tag{3.40}$$

corresponding to the particular cases of the superposition symmetry (1.8), namely,

$$S_1 = \frac{\partial}{\partial \boldsymbol{E}}\,, \quad S_2 = \frac{\partial}{\partial \boldsymbol{H}}\,\cdot$$

Example 11.6. Substituting in (3.39) the travelling wave solution

$$\boldsymbol{E}_* = \begin{pmatrix} 0 \\ f(x-t) \\ g(x-t) \end{pmatrix}, \quad \boldsymbol{H}_* = \begin{pmatrix} 0 \\ -g(x-t) \\ f(x-t) \end{pmatrix}$$

of Eqs. (1.1)-(1.2) we obtain the conservation law

$$\frac{d}{dt}\int \big\{f(x-t)\,[E^3 - H^2] - g(x-t)\,[E^2 + H^3]\big\}\, dxdydz = 0 \tag{3.41}$$

with two arbitrary functions, $f(x-t)$ and $g(x-t)$.

4 Conservation laws for Eqs. (1.1)–(1.2)

4.1 Splitting of conservation law (3.31) by Eqs. (1.2)

Comparing τ_{12} given by Eq. (3.27) with σ_3 given in (1.42), and the flux components (3.29) with the quantities

$$\chi_3^1 = -\big(E^1E^3 + H^1H^3\big), \quad \chi_3^2 = -\big(E^2E^3 + H^2H^3\big),$$
$$\chi_3^3 = \frac{1}{2}\big[(E^1)^2 + (E^2)^2 - (E^3)^2 + (H^1)^2 + (H^2)^2 - (H^3)^2\big]$$

given by Eqs. (1.41), we conclude that the conservation equation with the density (3.27) and the flux (3.29) satisfied for for Eqs. (1.1) splits into two conservation equations for Eqs. (1.1)-(1.2). One of them is the conservation equation (1.43) for the linear momentum with the density $\sigma = \boldsymbol{E} \times \boldsymbol{H}$. The other has the density

$$\tau_* = y\pi_1 - x\pi_2 \tag{4.1}$$

and the flux $\boldsymbol{\chi}_*$ with the components

$$\begin{aligned}
\chi_*^1 = E^1E^3 + H^1H^3 &+ x\big(E^3E_y^2 - E^2E_y^3 + H^3H_y^2 - H^2H_y^3\big)\\
&- y\big(E^3E_x^2 - E^2E_x^3 + H^3H_x^2 - H^2H_x^3\big),\\
\chi_*^2 = E^2E^3 + H^2H^3 &+ x\big(E^1E_y^3 - E^3E_y^1 + H^1H_y^3 - H^3H_y^1\big)\\
&- y\big(E^1E_x^3 - E^3E_x^1 + H^1H_x^3 - H^3H_x^1\big),\\
\chi_*^3 = \big(E^3\big)^2 + \big(H^3\big)^2 &+ x\big(E^2E_y^1 - E^1E_y^2 + H^2H_y^1 - H^1H_y^2\big)\\
&- y\big(E^2E_x^1 - E^1E_x^2 + H^2H_x^1 - H^1H_x^2\big).
\end{aligned} \tag{4.2}$$

Proceeding likewise with all components of the vector valued conservation equation (3.31) we arrive at the following result.

Lemma 11.7. The equations (1.2) of Maxwell's system split the conservation equation (3.31) into two equations. In consequence, the group of rotations generated by X_{12}, X_{13}, X_{23} from (1.3) provide *two* vector valued conservation laws for the Maxwell equations (1.1)-(1.2), namely (cf. the linear momentum (1.44)):

$$\frac{d}{dt}\int \big(\boldsymbol{E} \times \boldsymbol{H}\big)\, dxdydz = 0 \tag{4.3}$$

and

$$\frac{d}{dt}\int \big(\boldsymbol{x} \times \boldsymbol{\pi}\big)\, dxdydz = 0, \tag{4.4}$$

where $\boldsymbol{\pi} = (\pi_1, \pi_2, \pi_3)$ is the vector with the components (3.23).

4.2 Conservation laws due to Lorentz symmetry

Acting by the operator (2.10) on the Lagrangian (2.4) and using Eqs. (2.11) we obtain:

$$X_{01}(\mathcal{L})\big|_{(1.1)} = -E^1(\nabla \cdot \boldsymbol{H}) + H^1(\nabla \cdot \boldsymbol{E}).$$

Therefore the operator X_{01} satisfies the invariance test (3.8) for the solutions of the Maxwell equations (1.1)–(1.2):

$$X_{01}(\mathcal{L})\Big|_{(1.1)-(1.2)} = 0.$$

This is true for all generators $X_{0i},\ i = 1, 2, 3$, (see (1.4)) of the Lorentz transformations. Consequently, the conservation laws associated with the Lorentz transformations can be computed by means of Eqs. (3.10). I will calculate here only the densities τ_{0i} of these conservation laws.

Let us consider X_{01}. Eqs. (3.10) yield:

$$\begin{aligned}\tau_{01} &= (tE^1_x + xE^1_t)H^1 - (H^3 - tE^2_x - xE^2_t)H^2 + (H^2 + tE^3_x + xE^3_t)H^3 \\ &\quad - (tH^1_x + xH^1_t)E^1 - (E^3 + tH^2_x + xH^2_t)E^2 + (E^2 - tH^3_x - xH^3_t)E^3.\end{aligned}$$

We rewrite it in the form

$$\begin{aligned}\tau_{01} &= x\left(H^1E^1_t + H^2E^2_t + H^3E^3_t - E^1H^1_t - E^2H^2_t - E^2H^3_t\right) \\ &\quad + t\left(H^1E^1_x + H^2E^2_x + H^3E^3_x - E^1H^1_x - E^2H^2_x - E^2H^3_x\right) \\ &= x\left(\boldsymbol{H}\cdot\boldsymbol{E}_t - \boldsymbol{E}\cdot\boldsymbol{H}_t\right) + t\left(\boldsymbol{H}\cdot\boldsymbol{E}_x - \boldsymbol{E}\cdot\boldsymbol{H}_x\right).\end{aligned}$$

Finally, using the conservation densities π_0 and π_1 given by Eqs. (3.13) and (3.23), respectively, we obtain:

$$\tau_{01} = x\pi_0 - t\pi_1.$$

Proceeding likewise with all generators X_{0i} from (1.4), we arrive at the following.

Lemma 11.8. The Lorentz invariance of Maxwell's equations (1.1)–(1.2) provides the vector valued conservation law

$$\frac{d}{dt}\int(\pi_0\boldsymbol{x} - t\boldsymbol{\pi})\,dxdydz = 0, \tag{4.5}$$

where

$$\boldsymbol{\pi} = (\pi_1, \pi_2, \pi_3) \tag{4.6}$$

is the vector with the components (3.23).

4.3 Conservation laws due to conformal symmetry

Acting by the operator Y_1 from (1.9) on the Lagrangian (2.4) and using Eqs. (2.13) we see that the invariance test (3.8) is satisfied:

$$Y_1(\mathcal{L})\Big|_{(1.1)-(1.2)} = 0.$$

Therefore the conservation laws associated with the conformal transformations can be computed by means of Eqs. (3.10). Let us calculate the density τ of the conservation law provided by the operator Y_1. Eqs. (3.10) yield:

$$\begin{aligned}\tau =& \big(\eta^\alpha - \xi^j u_j^\alpha\big)\frac{\partial \mathcal{L}}{\partial u_t^\alpha} = -\Big[4xE^1 + 2yE^2 + 2zE^3 + (x^2 - y^2 - z^2 + t^2)E_x^1 \\ &+ 2xyE_y^1 + 2xzE_z^1 + 2xtE_t^1\Big]\frac{\partial \mathcal{L}}{\partial E_t^1} - \cdots - \Big[4xH^3 - 2zH^1 - 2tE^2 \\ &+ (x^2 - y^2 - z^2 + t^2)H_x^3 + 2xyH_y^3 + 2xzH_z^3 + 2xtH_t^3\Big]\frac{\partial \mathcal{L}}{\partial H_t^3}\,.\end{aligned}$$

Substituting

$$\frac{\partial \mathcal{L}}{\partial E_t^1} = -H^1, \ldots, \frac{\partial \mathcal{L}}{\partial H_t^3} = E^3$$

and arranging in terms of different powers of x, y, z, t, we have:

$$\begin{aligned}\tau =& \ 4y\big(E^2H^1 - E^1H^2\big) + 4z\big(E^3H^1 - E^1H^3\big) \\ &+ (x^2 - y^2 - z^2 + t^2)\left(H^1E_x^1 + H^2E_x^2 + H^3E_x^3 - E^1H_x^1 - E^2H_x^2 - E^2H_x^3\right) \\ &+ 2xy\left(H^1E_y^1 + H^2E_y^2 + H^3E_y^3 - E^1H_y^1 - E^2H_y^2 - E^2H_y^3\right) \\ &+ 2xz\left(H^1E_z^1 + H^2E_z^2 + H^3E_z^3 - E^1H_z^1 - E^2H_z^2 - E^2H_z^3\right) \\ &+ 2xt\left(H^1E_t^1 + H^2E_t^2 + H^3E_t^3 - E^1H_t^1 - E^2H_t^2 - E^2H_t^3\right),\end{aligned}$$

or

$$\begin{aligned}\tau =& -4y\big(\boldsymbol{E}\times\boldsymbol{H}\big)^3 + 4z\big(\boldsymbol{E}\times\boldsymbol{H}\big)^2 \\ &+ (x^2 - y^2 - z^2 + t^2)\left(\boldsymbol{H}\cdot\boldsymbol{E}_x - \boldsymbol{E}\cdot\boldsymbol{H}_x\right) + 2xy\left(\boldsymbol{H}\cdot\boldsymbol{E}_y - \boldsymbol{E}\cdot\boldsymbol{H}_y\right) \\ &+ 2xz\left(\boldsymbol{H}\cdot\boldsymbol{E}_z - \boldsymbol{E}\cdot\boldsymbol{H}_z\right) + 2xt\left(\boldsymbol{H}\cdot\boldsymbol{E}_t - \boldsymbol{E}\cdot\boldsymbol{H}_t\right).\end{aligned}$$

Finally, using the quantities π_0 and π_1, π_2, π_3 given by Eq. (3.13) and Eqs. (3.23), respectively, we write the resulting conservation density in the form

$$\begin{aligned}\tau = &- 4\big[\boldsymbol{x} \times \big(\boldsymbol{E} \times \boldsymbol{H}\big)\big]^1 + (t^2 - x^2 - y^2 - z^2)\pi_1 \\ &+ 2x(x\pi_1 + y\pi_2 + z\pi_3 + t\pi_0).\end{aligned} \tag{4.7}$$

Using the test for conservation densities (Section 1.5) one can verify that the terms of (4.7) that are linear in the independent variables and those quadratic in these variables provide two independent conservation densities (cf. Example 11.4). Treating likewise the operators Y_2 and Y_3 we arrive at the following result.

Lemma 11.9. The invariance of Maxwell's equations (1.1)–(1.2) with respect to the conformal transformations generated by Y_1, Y_2, Y_3 provides two vector valued conservation laws. Namely, the conservation of the angular momentum (1.46):

$$\frac{d}{dt}\int \big[\boldsymbol{x} \times (\boldsymbol{E} \times \boldsymbol{H})\big] dxdydz = 0, \tag{4.8}$$

and the following new vector valued conservation law:

$$\frac{d}{dt}\int \big[2(\boldsymbol{x} \cdot \boldsymbol{\pi} + t\pi_0)\boldsymbol{x} + (t^2 - x^2 - y^2 - z^2)\boldsymbol{\pi}\big] dxdydz = 0, \tag{4.9}$$

where

$$\boldsymbol{x} = (x, y, z), \;\; \boldsymbol{\pi} = (\pi_1, \pi_2, \pi_3).$$

The reckoning shows that the operator Y_4 from (1.9) leads to the following conservation density instead of (4.7):

$$\tau = 2t(x\pi_1 + y\pi_2 + z\pi_3) + (t^2 + x^2 + y^2 + z^2)\pi_0. \tag{4.10}$$

Hence, we have the following result.

Lemma 11.10. The invariance of Maxwell's equations (1.1)–(1.2) with respect to the conformal transformations generated by Y_4 provides the conservation law

$$\frac{d}{dt}\int \big[2t(\boldsymbol{x} \cdot \boldsymbol{\pi}) + (t^2 + x^2 + y^2 + z^2)\pi_0\big] dxdydz = 0. \tag{4.11}$$

5 Summary

It is exhibited that the Noether theorem can be applied to overdetermined systems of differential equations, provided that the system in question contains a sub-system admitting a variational formulation. In the case of the Maxwell equations the appropriate sub-system is provided by the evolution equations (1.1) of Maxwell's system. Using the Lagrangian (2.4) of this sub-system, one obtains an infinite set of conservation laws containing the conservation of the classical electromagnetic energy, linear and angular momenta, as well as Lipkin's conservation laws. This set of conservation laws does not contain, however, the relativistic center-of-mass theorem and the five conservation laws associated by Bessel-Hagen with the conformal transformations.

Tests for conservation densities are proved in Sections 1.4 and 1.5.

5.1 Conservation laws

The results on conservation laws obtained in the previous sections are summarized in the following theorem. For the sake of brevity, the conservation equations are written in the integral form.

Theorem 11.4. The symmetries (1.3)–(1.9) applied to the Lagrangian (2.4) of the evolutionary part (1.1) of Maxwell's vacuum equations provide the following conservation laws for the Maxwell equations (1.1)–(1.2):

Classical conservation laws:

$$\frac{d}{dt}\int \big(|\boldsymbol{E}|^2 + |\boldsymbol{H}|^2\big)dxdydz = 0 \quad \text{(energy conservation)}, \tag{5.1}$$

$$\frac{d}{dt}\int \big(\boldsymbol{E}\times\boldsymbol{H}\big)\,dxdydz = 0 \qquad \text{(linear momentum)}, \tag{5.2}$$

$$\frac{d}{dt}\int \big[\boldsymbol{x}\times(\boldsymbol{E}\times\boldsymbol{H})\big]dxdydz = 0 \quad \text{(angular momentum)}. \tag{5.3}$$

Non-classical conservation laws:

$$\frac{d}{dt}\int \pi_0\, dxdydz = 0, \tag{5.4}$$

$$\frac{d}{dt}\int \boldsymbol{\pi}\, dxdydz, \tag{5.5}$$

$$\frac{d}{dt}\int \big(\boldsymbol{x}\times\boldsymbol{\pi}\big)\, dxdydz = 0, \tag{5.6}$$

$$\frac{d}{dt}\int \big(t\pi_0 + \boldsymbol{x}\cdot\boldsymbol{\pi}\big)\, dxdydz = 0, \tag{5.7}$$

$$\frac{d}{dt}\int (\pi_0\boldsymbol{x} - t\boldsymbol{\pi})\, dxdydz = 0, \tag{5.8}$$

$$\frac{d}{dt}\int \big[2(\boldsymbol{x}\cdot\boldsymbol{\pi} + t\pi_0)\boldsymbol{x} + (t^2 - x^2 - y^2 - z^2)\boldsymbol{\pi}\big] dxdydz = 0, \tag{5.9}$$

$$\frac{d}{dt}\int \big[2t(\boldsymbol{x}\cdot\boldsymbol{\pi}) + (t^2 + x^2 + y^2 + z^2)\pi_0\big] dxdydz = 0, \tag{5.10}$$

$$\frac{d}{dt}\int \big(\boldsymbol{H}_*\cdot\boldsymbol{E} - \boldsymbol{E}_*\cdot\boldsymbol{H}\big)\, dxdydz = 0. \tag{5.11}$$

In Eqs. (5.4)–(5.10), π_0 is given by Eq. (3.13) and $\boldsymbol{\pi} = (\pi_1, \pi_2, \pi_3)$ is the vector with the components (3.23). The infinite set of conservation equations (5.11) involves two arbitrary solutions, $(\boldsymbol{E}, \boldsymbol{H})$ and $(\boldsymbol{E}_*, \boldsymbol{H}_*)$, of the Maxwell equations (1.1)–(1.2).

Remark 11.9. If the solution $(\boldsymbol{E}, \boldsymbol{H})$ is identical with the solution $(\boldsymbol{E}_*, \boldsymbol{H}_*)$, then the conservation law (5.11) is trivial, i.e. its density

$$\boldsymbol{H}_*\cdot\boldsymbol{E} - \boldsymbol{E}_*\cdot\boldsymbol{H}$$

vanishes. This is in accordance with the fact that in the case

$$\boldsymbol{E}_* = \boldsymbol{E}, \quad \boldsymbol{H}_* = \boldsymbol{H}$$

the superposition generator S coincides with the dilation generator Z_1 which does not provide a nontrivial conservation law (see Section 3.6).

5.2 Symmetries associated with conservation laws

The correspondence between the conservation laws summarized in Section 5.1 and the symmetries (1.3)-(1.9) of the Maxwell equations is given in the following table.

Conservation law	Symmetry
Energy conservation (5.1)	Duality symmetry Z_0 (1.5)
Linear momentum (5.2)	Rotation symmetries X_{12}, X_{13}, X_{23} (1.3)
Angular momentum (5.3)	Conformal symmetries Y_1, Y_2, Y_3 from (1.9)
Conservation law (5.4)	Time translation X_0 from (1.3)
Conservation law (5.5)	Spatial translations X_1, X_2, X_3 from (1.3)
Conservation law (5.6)	Rotation symmetries X_{12}, X_{13}, X_{23} (1.3)
Conservation law (5.7)	Space-time dilation Z_2 (1.7)
Conservation law (5.8)	Lorentz symmetries X_{01}, X_{02}, X_{03} (1.4)
Conservation law (5.9)	Conformal symmetries Y_1, Y_2, Y_3 from (1.9)
Conservation law (5.10)	Conformal symmetry Y_4 from (1.9)
Conservation law (5.11)	Superposition symmetry S (1.8)

Paper 12

Quasi-self-adjoint equations

Unabridged draft of the paper [51]

1 Adjoint equations

Consider a system of m differential equations (linear or non-linear)

$$F_\alpha\big(x, u, u_{(1)}, \ldots, u_{(s)}\big) = 0, \quad \alpha = 1, \ldots, m, \tag{1.1}$$

with m dependent variables

$$u = (u^1, \ldots, u^m)$$

and n independent variables

$$x = (x^1, \ldots, x^n).$$

Eqs. (1.1) involve first-order partial derivatives

$$u_{(1)} = \{u_i^\alpha\}$$

and higher-order derivatives up to sth-order derivatives $u_{(s)}$.

The *adjoint equations* to Eqs. (1.1) are defined in [46] by

$$F_\alpha^*\big(x, u, v, \ldots, u_{(s)}, v_{(s)}\big) = 0, \quad \alpha = 1, \ldots, m, \tag{1.2}$$

where

$$F_\alpha^*\big(x, u, v, \ldots, u_{(s)}, v_{(s)}\big) = \frac{\delta(v^\beta F_\beta)}{\delta u^\alpha}\,\cdot \tag{1.3}$$

Here $v = (v^1, \ldots, v^m)$ are new dependent variables, $v_{(1)}, \ldots, v_{(s)}$ are their derivatives, $v^\beta F_\beta = \sum_{\beta=1}^m v^\beta F_\beta$, and $\delta/\delta u^\alpha$ is the Euler-Lagrange operator for the variational derivatives:

$$\frac{\delta(v^\beta F_\beta)}{\delta u^\alpha} = \frac{\partial(v^\beta F_\beta)}{\partial u^\alpha} - D_i\left(\frac{\partial(v^\beta F_\beta)}{\partial u_i^\alpha}\right) + D_i D_k\left(\frac{\partial(v^\beta F_\beta)}{\partial u_{ik}^\alpha}\right) - \cdots .$$

2 Self-adjointness

The system (1.1) is said to be *self-adjoint*[46] if the system of adjoint equations (1.2) becomes equivalent to the original system (1.1) upon the substitution

$$v = u. \tag{2.1}$$

It means that the following equations are satisfied

$$F_\alpha^*(x, u, u, \ldots, u_{(s)}, u_{(s)}) = \lambda_\alpha^\beta\, F_\beta(x, u, \ldots, u_{(s)}), \quad \alpha = 1, \ldots, m, \tag{2.2}$$

with undetermined coefficients λ_α^β. In general, the coefficients λ_α^β may be variable. This definition of self-adjoint equations was motivated by the fact that if a *linear* equation $L(u) = 0$ is self-adjoint, then the adjoint equation $L_*(v) = 0$ coincides with the original equation after substituting $v = u$, namely, $L_*(u) = L(u)$.

Remark 12.1. Self-adjoint equations have remarkable properties. For example, the "nonlocal" variables v can be eliminated from conservation laws. This important fact was applied in [45] to the Korteweg-de Vries equation.

The following examples illustrate the approach to determining self-adjoint equations. They are taken from [45].

Example 12.1. Consider the second-order evolution equations of the form

$$u_t = f(u)u_{xx}. \tag{2.3}$$

According to Eq. (1.3), we have:

$$\begin{aligned}\frac{\delta}{\delta u}\Big[(u_t - f(u)u_{xx})v\Big] &= -v_t - f'(u)vu_{xx} - D_x^2\big(vf(u)\big)\\ &= -v_t - f'(u)vu_{xx} - D_x\big(f(u)v_x + vf'(u)u_x\big)\\ &= -v_t - 2f'(u)vu_{xx} - f(u)v_{xx} - 2f'(u)u_xv_x - vf''(u)u_x^2.\end{aligned}$$

Therefore, the adjoint equation (1.2) to Eq. (2.3) is

$$v_t = -2f'(u)vu_{xx} - f(u)v_{xx} - 2f'(u)u_xv_x - vf''(u)u_x^2. \tag{2.4}$$

Upon setting $v = u$ it becomes:

$$u_t = -\big[2uf'(u) + f(u)\big]u_{xx} - \big[2f'(u) + uf''(u)\big]u_x^2. \tag{2.5}$$

In our case Eq. (2.2) is written:

$$u_t + \big[2uf'(u) + f(u)\big]u_{xx} + \big[2f'(u) + uf''(u)\big]u_x^2 = \lambda u_t - \lambda f(u)u_{xx}, \tag{2.6}$$

whence $\lambda = 1$ and

$$2uf'(u) + f(u) = -f(u), \quad 2f'(u) + uf''(u) = 0. \tag{2.7}$$

The second equation in (2.7) is the differential consequence of the first one. Therefore, we solve the first equation in (2.7),

$$uf'(u) + f(u) = 0,$$

and obtain

$$f(u) = \frac{a}{u}\,, \quad a = \text{const}.$$

Hence, Eq. (2.3) is self-adjoint if and only if it has the form

$$u_t = \frac{a}{u}\,u_{xx}, \quad a = \text{const}. \tag{2.8}$$

Example 12.2. Let us find all self-adjoint equations among the equations

$$u_t = f(u)u_{xxx}. \tag{2.9}$$

We have:

$$\frac{\delta}{\delta u}\Big[(u_t - f(u)u_{xxx})v\Big] = -v_t - f'(u)vu_{xxx} + D_x^3\big[vf(u)\big]$$
$$= -v_t + fv_{xxx} + 3\big[f'v_x + f''vu_x\big]u_{xx} + 3f'u_xv_{xx} + 3f''v_xu_x^2 + f'''vu_x^3.$$

Hence, the adjoint equation to Eq. (2.9) is

$$v_t = fv_{xxx} + 3\big[f'v_x + f''vu_x\big]u_{xx} + 3f'u_xv_{xx} + 3f''v_xu_x^2 + f'''vu_x^3. \tag{2.10}$$

Letting in (2.10) $v = u$, we have:

$$u_t = f u_{xxx} + 3(2f' + uf'')u_x u_{xx} + (3f'' + uf''')u_x^3 = 0. \tag{2.11}$$

Comparison of Eqs. (2.11) and (2.9) yields the system

$$2f' + uf'' = 0, \quad 3f'' + uf''' = 0.$$

Since the second equation of this system is obtained from the first one by differentiation, we integrate the equation

$$2f' + uf'' = 0$$

and obtain:

$$f(u) = \frac{a}{u} + b, \quad a, b = \text{const}.$$

Hence, the general self-adjoint equation of the form (2.9) is

$$u_t = \Big(\frac{a}{u} + b\Big) u_{xxx}, \quad a, b = \text{const}. \tag{2.12}$$

These examples show that, e.g. the nonlinear equations

$$u_t = u^2 u_{xx} \tag{2.13}$$

and

$$u_t = u^3 u_{xxx} \tag{2.14}$$

having remarkable symmetry and physical properties (see, e.g. [99] and [34], Section 20) are not self-adjoint. Therefore one cannot eliminate the nonlocal variables from conservation laws of these equations by setting $v = u$.

I will generalize the concept of self-adjoint equations by introducing the notion of *quasi-self-adjoint equations.* Then Eqs. (2.13) and (2.14) will be self-adjoint in the generalized meaning.

3 Quasi-self-adjoint equations

Definition 12.1. The system (1.1) is said to be quasi-self-adjoint if the adjoint system (1.2) is equivalent to the system (1.1) upon the substitution

$$v = \varphi(u) \tag{3.1}$$

with a certain function $\varphi(u)$ such that $\varphi'(u) \neq 0$.

Example 12.3. Consider again Eq. (2.3) from Example 12.1:

$$u_t = f(u)u_{xx}, \quad f'(u) \neq 0. \tag{3.2}$$

We substitute

$$v = \varphi(u),\ v_t = \varphi' u_t,\ v_x = \varphi' u_x,\ v_{xx} = \varphi' u_{xx} + \varphi'' u_x^2$$

in the adjoint equation (2.4) and obtain:

$$\varphi' u_t + 2f'\varphi u_{xx} + \left[\varphi' u_{xx} + \varphi'' u_x^2\right] f + 2f'\varphi' u_x^2 + \varphi f'' u_x^2 = 0. \tag{3.3}$$

Eq. (3.3) is equivalent to Eq. (3.2) if (see Eq. (2.2))

$$\varphi' u_t + 2f'\varphi u_{xx} + \left[\varphi' u_{xx} + \varphi'' u_x^2\right] f + 2f'\varphi' u_x^2 + \varphi f'' u_x^2 = \lambda u_t - \lambda f u_{xx},$$

whence $\lambda = \varphi'$ and

$$2f'\varphi + f\varphi' = -f\varphi', \quad f\varphi'' + 2f'\varphi' + \varphi f'' = 0. \tag{3.4}$$

The second equation in (3.4) is the differential consequence of the first one. The first equation in (3.4) is written

$$f'\varphi + f\varphi' \equiv (f\varphi)' = 0$$

and yields

$$\varphi(u) = \frac{a}{f(u)}\,, \quad a = \text{const.}$$

We can take $a = 1$. Hence, Eq. (3.2) is quasi-self-adjoint for any function $f(u)$. Namely, the adjoint equation (2.4) becomes equivalent to the original equation (3.2) upon the substitution

$$v = \frac{1}{f(u)}\,\cdot \tag{3.5}$$

In particular, the adjoint equation $v_t + 4uvu_{xx} + u^2 v_{xx} + 4uu_x v_x + 2vu_x^2 = 0$ to Eq. (2.13), $u_t = u^2 u_{xx}$, is mapped to the original equation Eq. (2.13) by the substitution $v = u^{-2}$.

Example 12.4. Proceeding as above, one can verify that the equation

$$u_t = f(u)u_{xxx} \quad f'(u) \neq 0, \tag{3.6}$$

is quasi-self-adjoint for any function $f(u)$. The adjoint equation (2.10) becomes equivalent to the original equation (3.6) upon the substitution

$$v = \frac{1}{f(u)}\,\cdot \tag{3.7}$$

In particular, the adjoint equation to Eq. (2.14), $u_t = u^3 u_{xxx}$, is

$$v_t = u^3 v_{xxx} + 9\big[u^2 v_x + 2uvu_x\big]u_{xx} + 9u^2 u_x v_{xx} + 18uv_x u_x^2 + 6vu_x^3.$$

It is mapped into Eq. (2.14) by the substitution $v = u^{-3}$.

Example 12.5. The following simple approach allows one to obtain the more general result. Consider the equation

$$u_t = f(u)u_{(n)} \quad f'(u) \neq 0, \tag{3.8}$$

where $u_{(1)} = u_x,\ u_{(2)} = u_{xx},\ u_{(3)} = u_{xxx},\ u_{(4)} = u_{xxxx}$, etc. If we write the adjoint equation in the form

$$\frac{\delta}{\delta u}\Big[v(u_t - f(u)u_{(n)})\Big] = -v_t - f'(u)vu_{(n)} - \big(-D_x\big)^n\big(vf(u)\big) = 0$$

we see that it is becomes identical with Eq. (3.8) upon setting

$$v = \frac{1}{f(u)}\,\cdot \tag{3.9}$$

Indeed, then $vf(u) = 1$ and hence $\big(-D_x\big)^n\big(vf(u)\big) = 0$. Furthermore

$$-v_t - f'(u)vu_{(n)} = \frac{f'(u)}{f^2(u)}\,u_t - \frac{f'(u)}{f(u)}\,u_{(n)} = \frac{f'(u)}{f^2(u)}\,\big[u_t - f(u)u_{(n)}\big].$$

Remark 12.2. Any equation

$$u_t = u_{(n)} + F(x, u, u_{(1)}, \ldots, u_{(n-1)}) \tag{3.10}$$

with even n, in particular the heat equation, is not quasi-self-adjoint. It is manifest from the observation that the substitution (3.1) yields

$$v_t = \varphi'(u)u_t, \quad v_{(n)} = \varphi'(u)u_{(n)} + \cdots,$$

and hence after the substitution (3.1) we will have:

$$\frac{\delta}{\delta u}\Big[v\big(u_t - u_{(n)} - F\big)\Big] = -v_t - v_{(n)} + \cdots = -\varphi'(u)\big[u_t + u_{(n)}\big] + \cdots.$$

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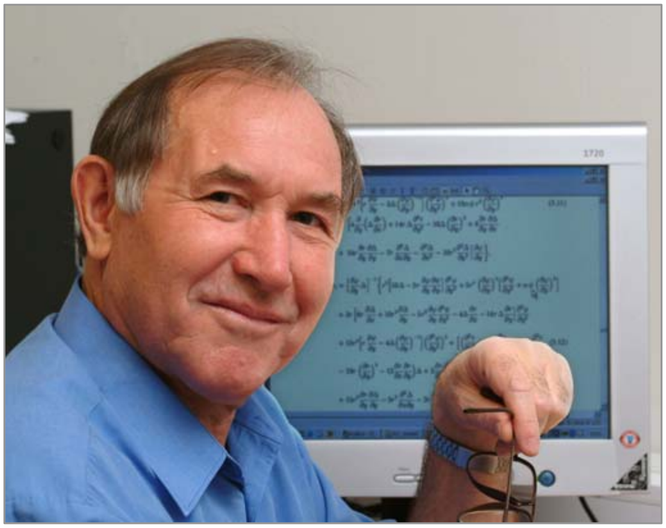

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